Seminar on Triangulated Categories

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1 Foundations

Our additive categories have finite direct sums.

Definition 1.1. The **automorphism** of an additive category C is an additive functor $T\colon \mathsf{C} \to \mathsf{C}$, which has inverse $T^{-1}\colon \mathsf{C} \to \mathsf{C}$. That is, $T^{-1}T = \mathrm{id}_{\mathsf{C}} = TT^{-1}$. The automorphism T is usually called the **translation functor**.

Definition 1.2. Let C be an additive category and T an automorphism of C. A **sextuple** (X, Y, Z, u, v, w) in C is given by objects $X, Y, Z \in \mathsf{C}$ and morphisms $u\colon X \to Y$, $v\colon Y \to Z$ and $w\colon Z \to TX$. A more suggestive notation of sextuples is: $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$.

Definition 1.3. A **morphism of sextuples** from (X, Y, Z, u, v, w) to (X', Y', Z', u', v', w') is a triple (f, g, h) of morphisms such that the following diagram commutes:

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Tf} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX'
\end{array}$$

If in this situation f, g and h are isomorphisms in C, the morphism is then called an **isomorphism**.

Definition 1.4. A set $\mathcal{T}$ of sextuples in C is called a **triangulation** of C if the following conditions are satisfied. The elements of $\mathcal{T}$ are then called **triangles**.

(TR1) Every sextuple isomorphic to a triangle is a triangle.

Every morphism $u\colon X \to Y$ in C can be embedded into a triangle (X, Y, Z, u, v, w).

The sextuple $(X, X, 0, \mathrm{id}_X, 0, 0)$ is a triangle. (id_X denotes the identity morphism from X to X).

(TR2) If (X, Y, Z, u, v, w) is a triangle, then $(Y, Z, TX, v, w, -Tu)$ is a triangle.

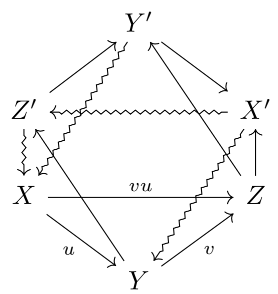
(TR3) Given two triangles (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') and morphisms $f\colon X \to X'$, $g\colon Y \to Y'$ such that $u'f = gu$, there exists a morphism (f, g, h) from the first triangle to the second.

(TR4) (The octahedral axiom) Consider triangles (X, Y, Z', u, i, i'), (Y, Z, X', v, j, j') and (X, Z, Y', vu, k, k'). Then there exist morphisms $f\colon Z' \to Y'$, $g\colon Y' \to X'$ such that the following diagram commutes and the third row is a triangle.

$$\begin{tikzcd}
T^{-1}Y' \arrow[r, "T^{-1}k'"] \arrow[d, "T^{-1}g"'] & X \arrow[r, equal] \arrow[d, "u"'] & X \arrow[d, "vu"'] & & \\
T^{-1}X' \arrow[r, "T^{-1}j"] & Y \arrow[r, "v"] \arrow[d, "i"'] & Z \arrow[r, "j"] \arrow[d, "k"'] & X' \arrow[r, "j'"] \arrow[d, equal] & TY \arrow[d, "Ti"'] \\
& Z' \arrow[r, "f"] \arrow[d, "i'"'] & Y' \arrow[r, "g"] \arrow[d, "k'"'] & X' \arrow[r, "j'Ti"] & TZ' \\
& TX \arrow[r, equal] & TX & &
\end{tikzcd}$$

The additive category C together with a translation functor T and a triangulation $\mathcal{T}$ is called a **triangulated category**.

Remark 1.1. A different way of displaying the axiom (TR4) is the following diagram:



Remark 1.2. The morphism h in (TR3) is NOT unique. For example, h can be arbitrary in the following diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"] & TX \arrow[r, "-\mathrm{id}_{TX}"] \arrow[d, "h"] & TX \arrow[d, "0"] \\
0 \arrow[r] & TY \arrow[r, "-\mathrm{id}_{TY}"] & TY \arrow[r] & 0
\end{tikzcd}$$

Definition 1.5. Let $\mathsf{C} = (\mathsf{C}, T, \mathcal{T})$, $\mathsf{C}' = (\mathsf{C}', T', \mathcal{T}')$ be triangulated categories. An additive functor $F\colon \mathsf{C} \to \mathsf{C}'$ is called **exact** if there exists an invertible natural transformation $\alpha\colon FT \to T'F$ such that $(FX, FY, FZ, Fu, Fv, \alpha_X Fw)$ is in $\mathcal{T}'$ whenever (X, Y, Z, u, v, w) is in $\mathcal{T}$.

Definition 1.6. If an exact functor $F\colon \mathsf{C} \to \mathsf{C}'$ is an equivalence of categories, we call it a **triangle-equivalence**. C and C' are then called **triangle-equivalent**.

Definition 1.7. An additive functor $H\colon \mathsf{C} \to \mathsf{A}$ from a triangulated category C to an abelian category A is called a **(covariant) cohomological functor**, if whenever (X, Y, Z, u, v, w) is a triangle, the long sequence

$$\cdots \longrightarrow H\left(T^i X\right) \xrightarrow{H\left(T^i u\right)} H\left(T^i Y\right) \xrightarrow{H\left(T^i v\right)} H\left(T^i Z\right) \xrightarrow{H\left(T^i w\right)} H\left(T^{i+1} X\right) \longrightarrow \cdots$$

is exact. The additive functor $H\colon \mathsf{C} \to \mathsf{A}$ is called a **(contravariant) cohomological functor**, if whenever (X, Y, Z, u, v, w) is a triangle, the long sequence

$$\cdots \longrightarrow H\left(T^{i+1} X\right) \xrightarrow{H\left(T^i w\right)} H\left(T^i Z\right) \xrightarrow{H\left(T^i v\right)} H\left(T^i Y\right) \xrightarrow{H\left(T^i u\right)} H\left(T^i X\right) \longrightarrow \cdots$$

is exact.

Proposition 1.1. Let C be a triangulated category, (X, Y, Z, u, v, w) be a triangle and M be an object in C. Then

(a) $vu = wv = (Tu)w = 0$.

(b) $\operatorname{Hom}_{\mathsf{C}}(M, -)$ and $\operatorname{Hom}_{\mathsf{C}}(-, M)$ are cohomological functors.

(c) Let (f, g, h) be a morphism of triangles from (X, Y, Z, u, v, w) to (X', Y', Z', u', v', w'). If f and g are isomorphisms, then so is h.

Proof. (a) In virtue of (TR2) it is enough to show that $vu = 0$. By (TR2) we know that $(Y, Z, TX, v, w, -Tu)$ is a triangle. By (TR3) the following diagram can be completed to a commutative diagram:

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "v"'] & Z \arrow[r, "w"] \arrow[d, equal] & TX \arrow[r, "-Tu"] & TY \arrow[d, "Tv"] \\
Z \arrow[r, equal] & Z \arrow[r, "0"] & 0 \arrow[r, "0"] & TZ
\end{tikzcd}$$

In particular $Tv(-Tu) = 0$, hence $vu = 0$.

(b) We show that $\operatorname{Hom}_{\mathsf{C}}(M, -)$ is exact. The other assertion follows dually. It is enough to show that

$$\operatorname{Hom}\left(M, T^i X\right) \xrightarrow{\operatorname{Hom}\left(M, T^i u\right)} \operatorname{Hom}\left(M, T^i Y\right) \xrightarrow{\operatorname{Hom}\left(M, T^i v\right)} \operatorname{Hom}\left(M, T^i Z\right)$$

is exact. From (a) we have $vu = 0$, so $T^i v T^i u = 0$, and so $\operatorname{Hom}\left(M, T^i v\right) \operatorname{Hom}\left(M, T^i u\right) = 0$. So let $g \in \operatorname{Hom}\left(M, T^i Y\right)$ such that $\left(T^i v\right) g = 0$. By (TR1) we obtain a triangle $\left(T^{-i} M, T^{-i} M, 0, \mathrm{id}_{T^{-i} M}, 0, 0\right)$. It follows that $\left(T^{-i} M, 0, T^{-i+1} M, 0, 0, -\mathrm{id}_{T^{-i+1} M}\right)$ is a triangle thanks to (TR2). Thus the rows of the following diagram are triangles

$$\begin{tikzcd}
T^{-i}M \arrow[r, "0"] \arrow[d, "T^{-i}g"'] & 0 \arrow[r, "0"] \arrow[d, "0"] & T^{-i+1}M \arrow[r, "-\mathrm{id}_{T^{-i+1}M}"] & T^{-i+1}M \arrow[d, "T^{-i+1}g"] \\
Y \arrow[r, "v"] & Z \arrow[r, "w"] & TX \arrow[r, "-Tu"] & TY
\end{tikzcd}$$

By assumption $v\left(T^{-i}g\right) = 0$. By (TR3) there exists h such that $(Tu)h = T^{-i+1}g$. Hence $\left(T^{i}u\right)T^{i-1}h = g$.

(c) Consider the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & TX \arrow[d, "Tf"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX'
\end{tikzcd}$$

and assume that f and g are isomorphism.

We apply the cohomological functor $\mathrm{Hom}(Z', -)$ and obtain the following commutative diagram which has exact rows by (b).

$$\begin{tikzcd}
\mathrm{Hom}(Z',X) \arrow[r] \arrow[d, "\mathrm{Hom}(Z',f)"'] & \mathrm{Hom}(Z',Y) \arrow[r] \arrow[d, "\mathrm{Hom}(Z',g)"] & \mathrm{Hom}(Z',Z) \arrow[r] \arrow[d, "\mathrm{Hom}(Z',h)"] & \mathrm{Hom}(Z',TX) \arrow[r] \arrow[d, "\mathrm{Hom}(Z',Tf)"] & \mathrm{Hom}(Z',TY) \arrow[d, "\mathrm{Hom}(Z',Tg)"] \\
\mathrm{Hom}(Z',X') \arrow[r] & \mathrm{Hom}(Z',Y') \arrow[r] & \mathrm{Hom}(Z',Z') \arrow[r] & \mathrm{Hom}(Z',TX') \arrow[r] & \mathrm{Hom}(Z',TY')
\end{tikzcd}$$

Since $\mathrm{Hom}(Z', f), \mathrm{Hom}(Z', g), \mathrm{Hom}(Z', Tf), \mathrm{Hom}(Z', Tg)$ are isomorphisms, we infer that $\mathrm{Hom}(Z', h)$ is an isomorphism from the Five Lemma.

In particular there exists $\varphi \in \mathrm{Hom}(Z', Z)$ such that $h\varphi = \mathrm{id}_{Z'}$. Conversely, apply the cohomological functor $\mathrm{Hom}(-, Z)$. As above we conclude that $\mathrm{Hom}(h, Z)$ is an isomorphism. In particular there exists $\psi \in \mathrm{Hom}(Z', Z)$ such that $h\psi = \mathrm{id}_Z$. Hence h is an isomorphism.

□

Corollary 1.1. In C, for any morphism $u\colon X \to Y$, it can only be embedded into a triangle unique up to an isomorphism. That is, if (X, Y, Z, u, v, w) and (X, Y, Z', u, v', w') are both triangles, then there exists an isomorphism $h\colon Z \to Z'$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "w"] \arrow[d, "h"] & TX \arrow[d, equal] \\
X \arrow[r, "u"] & Y \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX
\end{tikzcd}$$

Lemma 1.1. Let $F\colon \mathsf{C} \to \mathsf{C}'$ be a triangle-equivalence and let G be a quasi-inverse of F. Then G is a triangle-equivalence.

Proof. Since $F\colon \mathsf{C} \to \mathsf{C}'$ is a categorical equivalence, there exists $G\colon \mathsf{C}' \to \mathsf{C}$ and two natural isomorphisms $\eta\colon \mathrm{id}_{\mathsf{C}} \to GF$ and $\epsilon\colon FG \to \mathrm{id}_{\mathsf{C}'}$.

Let $\beta_{X'}\colon GT'X' \to TGX'$ defined as the composition

$$\beta_{X'} = \eta^{-1}_{TGX'} \circ G\left(\alpha^{-1}_{GX'}\right) \circ GT'\epsilon^{-1}_{X'}\colon GT'X' \to TGX'.$$

Clearly it is a natural isomorphism.

Now let (X', Y', Z', u', v', w') be a triangle in C'. By (TR1), there exists a triangle (GX', GY', Z, Gu', v, w). Since G is fully faithful and essentially surjective, there exists $W' \in \mathsf{C}'$ with $GW' = Z$. Since F is exact, $(FGX', FGY', FGW', FGu', Fv, \alpha_{GX'}Fw)$ is a triangle. Note that we have $FGX' \cong X'$, $FGY' \cong Y'$, $(FGX', FGY', FGW', FGu', Fv, \alpha_{GX'}Fw)$ is isomorphic to (X', Y', Z', u', v', w') thanks to Proposition 1.1 and (TR3). It follows that $FGW' \cong W' \cong Z'$.

$$\begin{tikzcd}
FGX' \arrow[r, "FGu'"] \arrow[d, "\epsilon_{X'}"'] & FGY' \arrow[r, "Fv"] \arrow[d, "\epsilon_{Y'}"'] & FGW' \arrow[r, "\alpha_{GX'}Fw"] \arrow[d, "h"'] & T'FGX' \arrow[d, "T\epsilon_{X'}"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & T'X'
\end{tikzcd}$$

Now we claim that $(GX', GY', GW', Gu', v, w)$ is isomorphic to $(GX', GY', GZ', Gu', Gv', \beta_{X'}Gw')$. From (TR2) $(GX', GY', GZ', Gu', Gv', \beta_{X'}Gw')$ is a triangle in C. It suffices to show the following diagram is commutative:

$$\begin{tikzcd}
GX' \arrow[r, "Gu'"] \arrow[d, equal] & GY' \arrow[r, "v"] \arrow[d, equal] & GW' \arrow[r, "w"] \arrow[d] & TGX' \arrow[d, equal] \\
GX' \arrow[r, "Gu'"] & GY' \arrow[r, "Gv'"] & GZ' \arrow[r, "\beta_{X'}Gw'"] & TGX'
\end{tikzcd}$$

where $GW' \to GZ'$ is defined as the composition $Gh \circ G\epsilon^{-1}_{W'}$.

(1) $\beta_{X'} \circ Gw' \circ Gh \circ G\epsilon^{-1}_{W'} = w$. Since $w' \circ h = T\epsilon_{X'} \circ \alpha_{GX'} \circ Fw$, we have

$$\begin{aligned}
\beta_{X'} \circ Gw' \circ Gh \circ G\epsilon^{-1}_{W'} &= \beta_{X'} \circ G\left(w' \circ h\right) \circ G\epsilon^{-1}_{W'} \\
&= \beta_{X'} \circ GT\epsilon_{X'} \circ G\alpha_{GX'} \circ GFw \circ G\epsilon^{-1}_{W'}.
\end{aligned}$$

From the definition of β,

$$\beta_{X'} \circ GT\epsilon_{X'} \circ G\alpha_{GX'} \circ GFw \circ G\epsilon^{-1}_{W'} = \eta^{-1}_{TGX'} \circ GFw \circ G\epsilon^{-1}_{W'}.$$

Note that $G\epsilon^{-1}_{W'} = \eta_{GW'}$, and η is natural, the composition is precisely w.

(2) $Gh \circ G\epsilon^{-1}_{W'} \circ v = Gv'$. Since $G\epsilon^{-1}_{W'} = \eta_{GW'}$ and η is natural,

$$Gh \circ G\epsilon^{-1}_{W'} \circ v = Gh \circ GFv \circ \eta_{GY'}$$

Then

$$Gh \circ GFv \circ \eta_{GY'} = Gh \circ GFv \circ G\epsilon_{Y'}^{-1}.$$

Since $h \circ Fv = v' \circ \epsilon_{Y'}$, we have $Gh \circ GFv = Gv' \circ G\epsilon_{Y'}$. It follows that

$$Gh \circ GFv \circ G\epsilon_{Y'}^{-1} = Gv'.$$

□

The following lemma shows that the converse of (TR2) is satisfied.

Lemma 1.2. If $(Y, Z, TX, v, w, -Tu)$ is a triangle, then (X, Y, Z, u, v, w) is a triangle.

Proof. If $(Y, Z, TX, v, w, -Tu)$ is a triangle we obtain by applying (TR2) that $(TX, TY, TZ, -Tu, -Tv, -Tw)$ is a triangle. By (TR1) there exists a triangle (X, Y, Z', u, v', w'), hence $(TX, TY, TZ', -Tu, -Tv', -Tw')$ is a triangle by (TR2), Now consider the following diagram:

$$\begin{tikzcd}
TX \arrow[r, "-Tu"] \arrow[d, equal] & TY \arrow[r, "-Tv"] \arrow[d, equal] & TZ \arrow[r, "-Tw"] \arrow[d, dashed, "h"] & T^2X \arrow[d, equal] \\
TX \arrow[r, "-Tu"] & TY \arrow[r, "-Tv'"] & TZ' \arrow[r, "-Tw'"] & T^2X
\end{tikzcd}$$

The morphism h exists by (TR3) which moreover is an isomorphism by Proposition 1.1 (c). So we obtain the following commutative diagram, where the second row is a triangle and the vertical morphisms are isomorphisms.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "w"] \arrow[d, "T^{-1}h"] & TX \arrow[d, equal] \\
X \arrow[r, "u"] & Y \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX
\end{tikzcd}$$

By (TR1) we infer that the first row is a triangle. □

Lemma 1.3. Let C be a triangulated category and (X, Y, Z, u, v, w) be a triangle. The following are equivalent:

(1) $w = 0$.

(2) u is a section.

(3) v is a retraction.

Proof. If $w = 0$ consider the following morphism of triangles. The existence of u' is guaranteed by (TR3).

$$\begin{tikzcd} Z \arrow[r, "w"] \arrow[d, "0"'] & TX \arrow[r, "-Tu"] \arrow[d, equal] & TY \arrow[r, "-Tv"] \arrow[d, "u'"] & TZ \arrow[d, "0"] \\ 0 \arrow[r, "0"] & TX \arrow[r, "-\mathrm{id}_{TX}"] & TX \arrow[r, "0"] & 0 \end{tikzcd}$$

Thus $(T^{-1}u')u = \mathrm{id}_X$, hence u is a section.

Conversely, if u is a section, there exists u' such that $(\mathrm{id}_X, u', 0)$ is a morphism from the triangle (X, Y, Z, u, v, w) to the triangle $(X, X, 0, \mathrm{id}_X, 0, 0)$. In particular $w \,\mathrm{id}_{TX} = 0$, hence $w = 0$.

In the same way one can show that (1) and (3) are equivalent. $\square$

Lemma 1.4. Let C be a triangulated category and (X, Y, Z, u, v, w) be a triangle. $f\colon W \to Z$ is a morphism. The following are equivalent:

(1) There exists $f'\colon W \to Z$ such that $vf' = f$.

(2) $wf = 0$.

Proof. By Proposition 1.1 (b) we know that $\mathrm{Hom}(W, Y) \xrightarrow{\mathrm{Hom}(W,v)} \mathrm{Hom}(W, Z) \xrightarrow{\mathrm{Hom}(W,w)} \mathrm{Hom}(W, TX)$ is exact. $\square$

Lemma 1.5. Let C be a triangulated category and (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') be triangles in C. Let $g\colon Y \to Y'$ be a morphism. Then the following are equivalent:

(1) $v'gu = 0$

(2) There exists a morphism (f, g, h) from the first triangle to the second.

If these conditions are satisfied and $\mathrm{Hom}\left(X, T^{-1}Z'\right) = 0$ then f and h are uniquely determined by g.

Proof. We apply $\mathrm{Hom}_{\mathsf{C}}(X, -)$ to the second triangle and obtain an exact sequence $\mathrm{Hom}(X, T^{-1}Z') \to \mathrm{Hom}(X, X') \to \mathrm{Hom}(X, Y') \to \mathrm{Hom}(X, Z')$. If $v'gu = 0$, there exists $f\colon X \to X'$ such that $u'f = gu$. By (TR3) we obtain (i)$\Longrightarrow$(ii).

Conversely, if (f, g, h) is a morphism then $v'gu = v'u'f = 0$ by Proposition 1.1.

We apply $\mathrm{Hom}_{\mathsf{C}}(X, -)$ to the second triangle and obtain an exact sequence $\mathrm{Hom}(X, T^{-1}Z') \to \mathrm{Hom}(X, X') \xrightarrow{\mathrm{Hom}(X,u')} \mathrm{Hom}(X, Y')$. If $\mathrm{Hom}(X, T^{-1}Z') = 0$, then $\mathrm{Hom}(X, u')$ is a monomorphism, which shows that f is uniquely determined by g. We apply $\mathrm{Hom}_{\mathsf{C}}(-, Z')$ to the second triangle and obtain an exact sequence $\mathrm{Hom}(TX,Z') \to \mathrm{Hom}(Z,Z') \xrightarrow{\mathrm{Hom}(v,Z')} \mathrm{Hom}(Y,Z')$. If $\mathrm{Hom}(X,T^{-1}Z')=0$, then also $\mathrm{Hom}(TX,Z')=0$. Thus $\mathrm{Hom}(v,Z')$ is a monomorphism, which shows that h is uniquely determined by g. □

Lemma 1.6. Let C be a triangulated category. Then $(X,Y,0,u,0,0)$ is a triangle if and only if u is an isomorphism.

Proof. If u is an isomorphism, then $(X,Y,0,u,0,0)$ is isomorphic to $(Y,Y,0,\mathrm{id}_Y,0,0)$. Thus the assertion follows from (TR1). If $(X,Y,0,u,0,0)$ is a triangle, we consider the following morphism of triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "u"] & Y \arrow[r, "0"] \arrow[d, equal] & 0 \arrow[r, "0"] \arrow[d, equal] & TX \arrow[d, "Tu"] \\ Y \arrow[r, "\mathrm{id}_Y"] & Y \arrow[r] & 0 \arrow[r] & TY \end{tikzcd}$$

Thus the assertion follows from Proposition 1.1. □

2 Frobenius categories

Definition 2.1. Let B be an additive category embedded as a full and extension-closed subcategory in some abelian category A. Let S be the set of exact sequences in A with terms in B. The pair (B,S) is called an **exact category**.

Definition 2.2. A morphism $u\colon X\to Y$ in B is called a **proper monomorphism** if there exists an exact sequence $0\to X\xrightarrow{u} Y\to Z\to 0$ in S. A morphism $v\colon Y\to Z$ in B is called a **proper epimorphism** if there exists an exact sequence $0\to X\to Y\xrightarrow{v} Z\to 0$ in S.

Definition 2.3. An object P in B is called **S-projective** if for all proper epimorphisms $v\colon Y\to Z$ and morphisms $f\colon P\to Z$ in B there exists $g\colon P\to Y$ such that $f=vg$.

An object I in B is called **S-injective** if for all proper monomorphisms $u\colon X\to Y$ and morphisms $f\colon X\to I$ in B there exists $g\colon Y\to I$ such that $f=gu$.

Lemma 2.1. Let (B,S) be an exact category. Then the following are equivalent:

1. $P\in\mathsf{B}$ is S-projective.

2. All exact sequences $0\to X\to Y\to P\to 0$ split.

Proof. First we assume that all exact sequences $0\to X\to Y\to P\to 0$ in S split. Let $v\colon Y\to Z$ be arbitrary proper epimorphism, i.e., there exists an exact sequence $0\to X\to Y\xrightarrow{v} Z\to 0$. $f\colon P\to Z$ is arbitrary morphism. Let W be the pullback of Y and P along the morphisms

$f\colon P \to Z$ and $v\colon Y \to Z$. Then we obtain an exact sequence $0 \to X \to W \xrightarrow{\pi_P} P \to 0$. Since $X, P \in \mathsf{B}$ and B is extension-closed, $W \in \mathsf{B}$. From the assumption, the exact sequence splits. So there exists $s\colon P \to W$ such that $\pi_P s = \mathrm{id}_P$. Therefore, $v(\pi_Y s) = f\pi_P s = f\,\mathrm{id}_P = f$. That is, P is S-projective.

Now if P is S-projective. Given arbitrary exact sequence $0 \to X \to Y \xrightarrow{v} P \to 0$, since v is proper epimorphism, there exists $g\colon P \to Y$ with $vg = \mathrm{id}_P$. So the above exact sequence is split. $\square$

Lemma 2.2. Let (B, S) be an exact category. Then the following are equivalent:

1. $I \in \mathsf{B}$ is S-injective.

2. All exact sequences $0 \to I \to Y \to Z \to 0$ in S split.

Definition 2.4. Let (B, S) be an exact category. We say that (B, S) has **enough S-projectives** if for all $Z \in \mathsf{B}$ there exists a proper epimorphism $v\colon P \to Z$ with P an S-projective in B. We say that (B, S) has **enough S-injectives** if for $X \in \mathsf{B}$ there exists a proper monomorphism $u\colon X \to I$ with I an S-injective in B.

Definition 2.5. An exact category (B, S) is called a **Frobenius category** if (B, S) has enough S-projectives and enough S-injectives and if moreover the S-projectives coincide with the S-injectives.

Definition 2.6. Let (B, S) be a Frobenius category. For a pair of objects X, Y in B we denote by $I(X, Y)$ the subgroup of morphisms from X to Y in B which factor over an S-injective. The **stable category** $\underline{\mathsf{B}}$ associated with B is the category whose objects are the objects of B and given two objects X, Y, the set of morphisms from X to Y is denoted by $\underline{\mathrm{Hom}}(X, Y)$, and $\underline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}_{\mathsf{B}}(X, Y)/I(X, Y)$. The residue class of a morphism $u\colon X \to Y$ is denoted by $\underline{u}$.

Lemma 2.3. Let $0 \to X \xrightarrow{\mu'} I' \xrightarrow{\pi'} X' \to 0$ and $0 \to X \xrightarrow{\mu''} I'' \xrightarrow{\pi''} X'' \to 0$ be in S such that I', I'' are S-injective. Then X' and X'' are isomorphic in $\underline{\mathsf{B}}$.

Proof. Since μ' and μ'' are proper monomorphisms and I' and I'' are S-injective we obtain the morphisms f, f' such that $f\mu' = \mu''$ and $f'\mu'' = \mu'$. Note that π' is cokernel of μ', and $(\pi'' f)\mu' = \pi''(f\mu') = \pi''\mu'' = 0$, so there exists a unique $g\colon X' \to X''$ such that $g\pi' = \pi'' f$ thanks to the universal property. Thus we obtain the following commutative diagram such that the rows belong to S.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\mu'"] \arrow[d, equal] & I' \arrow[r, "\pi'"] \arrow[d, "f"] & X' \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & X \arrow[r, "\mu''"] \arrow[d, equal] & I'' \arrow[r, "\pi''"] \arrow[d, "f'"] & X'' \arrow[r] \arrow[d, "g'"] & 0 \\
0 \arrow[r] & X \arrow[r, "\mu'"] & I' \arrow[r, "\pi'"] & X' \arrow[r] & 0
\end{tikzcd}$$

Thus $(f'f - \mathrm{id}_{I'})\,\mu' = 0$. So there exists $h\colon X' \to I'$ such that $h\pi' = f'f - \mathrm{id}_{I'}$ thanks to the universal property. Therefore $\pi' h\pi' = \pi'\,(f'f - \mathrm{id}_{I'}) = g'g\pi' - \pi' = (g'g - \mathrm{id}_{X'})\,\pi'$. Thus we have that $g'g - \mathrm{id}_{X'} = \pi' h$. In other words $\underline{g'g} = \underline{\mathrm{id}_{X'}}$. Similarly, we can show that $\underline{gg'} = \underline{\mathrm{id}_{X''}}$. $\square$

Let $X \in \mathsf{B}$. We denote by $[X]$ the isomorphism class of X in $\underline{\mathsf{B}}$. Moreover let $0 \to X \to I \to X' \to 0$ be in S such that I is S-injective. We assume that for all $X \in \mathsf{B}$ there is a bijection $\gamma_X\colon [X] \to [X']$. The preceding lemma shows that this assumption does not depend on the choice of $0 \to X \to I \to X' \to 0$.

For all $X \in \mathsf{B}$ we now choose elements $0 \to X \xrightarrow{\mu(X)} I(X) \xrightarrow{\pi(X)} TX \to 0$ in B with $I(X)$ an S-injective such that $TX = \gamma_X(X)$. Let $f\colon X \to Y$ be a morphism in B. Since $\mu(X)$ is a proper monomorphism and $I(Y)$ is S-injective we obtain a morphism $I(f)$ such that $\mu(Y)f = I(f)\mu(X)$. From the universal property we obtain a morphism $T(f)$ such that $T(f)\pi(X) = \pi(Y)I(f)$.

T is well-defined: Let $I_1(f), I_2(f)$ satisfies $I_i(f)\mu(X) = \mu(Y)f$ for $i = 1, 2$. Then $(I_1(f) - I_2(f))\mu(X) = 0$. Since $\pi(X)$ is the cokernel of $\mu(X)$, there exists unique $h\colon TX \to I(Y)$ such that $I_1(f) - I_2(f) = h\pi(X)$. Now $T_1(f), T_2(f)$ induced by $I_1(f), I_2(f)$ satisfies $T_i(f)\pi(X) = \pi(Y)I_i(f)$ for $i = 1, 2$. Therefore $(T_1(f) - T_2(f))\pi(X) = \pi(Y)(I_1(f) - I_2(f)) = \pi(Y)h\pi(X)$. Note that $\pi(X)$ is an epimorphism, $(T_1(f) - T_2(f)) = \pi(Y)h$. That is, $\underline{T_1(f)} = \underline{T_2(f)}$.

T is a functor: For $\mathrm{id}_X\colon X \to X$, we can choose $I(\mathrm{id}_X) = \mathrm{id}_{I(X)}$, then $T(\mathrm{id}_X)\pi(X) = \pi(X)\,\mathrm{id}_{I(X)} = \pi(X)$. From the uniqueness $T(\mathrm{id}_X) = \mathrm{id}_{TX}$. Then in $\underline{\mathsf{B}}$, $\underline{T(\mathrm{id}_X)} = \underline{\mathrm{id}_{TX}}$; Given $f\colon X \to Y$ and $g\colon Y \to Z$, we can choose $I(f), I(g)$ such that $I(f)\mu(X) = \mu(Y)f$ and $I(g)\mu(Y) = \mu(Z)g$. It follows that $(I(g)I(f))\mu(X) = I(g)\mu(Y)f = \mu(Z)gf$, so we may choose $I(gf) = I(g)I(f)$. Consequently

$$\begin{aligned}
T(gf)\pi(X) &= \pi(Z)I(gf) \\
&= \pi(Z)I(g)I(f) \\
&= T(g)\pi(Y)I(f) \\
&= T(g)T(f)\pi(X).
\end{aligned}$$

So $T(gf) = T(g)T(f)$. Then in $\underline{\mathsf{B}}$, $\underline{T(gf)} = \underline{T(g)T(f)}$.

T is additive: Given $f, f' \colon X \to Y$, similarly we may choose $I(f+f') = I(f) + I(f')$ because $(I(f)+I(f'))\mu(X) = \mu(Y)(f+f')$. Therefore

$$\begin{aligned} T(f+f')\pi(X) &= \pi(Y)I(f+f') \\ &= \pi(Y)(I(f)+I(f')) \\ &= \pi(Y)I(f) + \pi(Y)I(f') \\ &= T(f)\pi(X) + T(f')\pi(X) \\ &= (T(f)+T'(f))\pi(X). \end{aligned}$$

Then $T(f+f') = T(f) + T(f')$. So in $\underline{\mathsf{B}}$ we have $\underline{T(f+f')} = \underline{T(f)} + \underline{T(f')}$.

Proposition 2.1. T is an automorphism of $\underline{\mathsf{B}}$.

Proof. Let $\gamma_{TX}^{-1} \colon [TX] \to [X]$ be the inverse of the bijection $\gamma_X \colon [X] \to [TX]$. Let $T^{-1}(X) = \gamma_X^{-1}(X)$. Then $T^{-1} \colon \underline{\mathsf{B}} \to \underline{\mathsf{B}}$ is a functor, and clearly $T^{-1}T = TT^{-1} = \mathrm{id}_{\underline{\mathsf{B}}}$. □

Remark 2.1. Without the existence of a bijection γ_X as above, we still may choose for $X \in \mathsf{B}$ elements $0 \to X \to I(X) \to S(X) \to 0$ in S with $I(X)$ an S-injective. The same construction as above then yields that S may be considered as a functor on $\underline{\mathsf{B}}$. This time providing us with a self-equivalence. We claim that any two such choices yield isomorphic functors. In fact suppose that two such assignments have been made. So we have for all $X \in \mathsf{B}$ elements $0 \to X \xrightarrow{\mu(X)} I(X) \xrightarrow{\pi(X)} S(X) \to 0$ and $0 \to X \xrightarrow{\mu'(X)} I'(X) \xrightarrow{\pi'(X)} S'(X) \to 0$ in S such that $I(X)$ and $I'(X)$ are S-injective. Since $\mu(X)$ is a proper monomorphism and $I'(X)$ is S-injective we obtain $f_X \colon I(X) \to I'(X)$ such that $f_X\mu(X) = \mu'(X)$. This induces a morphism $\alpha(X) \colon S(X) \to S'(X)$ such that $\alpha(X)\pi(X) = \pi'(X)f_X$. Similarly, let $g_X \colon I'(X) \to I(X)$, we have $\beta(X) \colon S'(X) \to S(X)$ with $\beta(X)\pi'(X) = \pi(X)g_X$. Now we have

$$(\beta(X)\alpha(X))\pi(X) = \beta(X)\pi'(X)f_X = \pi(X)g_Xf_X.$$

Since

$$(g_Xf_X)\mu(X) = g_X\mu'(X) = \mu(X) = \mathrm{id}_{I(X)}\,\mu(X),$$

we have $\left(g_Xf_X - \mathrm{id}_{I(X)}\right)\mu(X) = 0$. From the universal property, there exists $h \colon S(X) \to I(X)$ such that $g_Xf_X - \mathrm{id}_{I(X)} = h \circ \pi(X)$. Thus

$$\pi(X)g_Xf_X = \pi(X)\left(\mathrm{id}_{I(X)} + h\pi(X)\right) = \pi(X) + \pi(X)h\pi(X),$$

which implies that

$$(\beta(X)\alpha(X))\pi(X) = \left(\mathrm{id}_{S(X)} + \pi(X)h\right)\pi(X),$$

and so $\beta(X)\alpha(X) = \mathrm{id}_{S(X)} + \pi(X)h$. That is $\underline{\beta(X)\alpha(X)} = \underline{\mathrm{id}_{S(X)}}$. Similarly $\underline{\alpha(X)\beta(X)} = \underline{\mathrm{id}_{S'(X)}}$. Therefore $\underline{\alpha(X)}$ is an isomorphism in $\underline{\mathsf{B}}$. Thus it remains to be seen that $\underline{\alpha(X)}$ is a natural transformation. In fact, let $f\colon X \to Y$. This induces the following two commutative diagrams:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
X \arrow[r, "f"] \arrow[d, "\mu(X)"'] & Y \arrow[d, "\mu(Y)"] \\
I(X) \arrow[r, "I(f)"] \arrow[d, "\pi(X)"'] & I(Y) \arrow[d, "\pi(Y)"] \\
S(X) \arrow[r, "S(f)"] \arrow[d] & S(Y) \arrow[d] \\
0 & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
X \arrow[r, "f"] \arrow[d, "\mu'(X)"'] & Y \arrow[d, "\mu'(Y)"] \\
I'(X) \arrow[r, "I'(f)"] \arrow[d, "\pi'(X)"'] & I'(Y) \arrow[d, "\pi'(Y)"] \\
S'(X) \arrow[r, "S'(f)"] \arrow[d] & S'(Y) \arrow[d] \\
0 & 0
\end{tikzcd}$$

We have morphisms $f_X\colon I(X) \to I'(X)$ and $f_Y\colon I(Y) \to I'(Y)$ such that $f_X\mu(X) = \mu'(X)$ and $f_Y\mu(Y) = \mu'(Y)$. Moreover we have morphisms $\alpha(X)\colon S(X) \to S'(X)$ and $\alpha(Y)\colon S(Y) \to S'(Y)$ such that $\alpha(X)\pi(X) = \pi'(X)f_X$ and $\alpha(Y)\pi(Y) = \pi'(Y)f_Y$. We claim that $\underline{\alpha(Y)S(f)} = \underline{S'(f)\alpha(X)}$. We show that $\alpha(Y)S(f) = S'(f)\alpha(X)$ factors over an S-injective. Observe that $(f_Y I(f) - I'(f)f_X)\,\mu(X) = 0$. Thus there exists $g\colon S(X) \to I'(Y)$ such that $g\pi(X) = f_Y I(f) - I'(f)f_X$. Now $\pi'(Y)g\pi(X) = \pi'(Y)\,(f_Y I(f) - I'(f)f_X) = \pi'(Y)f_Y I(f) - \pi'(Y)I'(f)f_X = \alpha(Y)\pi(Y)I(f) - S'(f)\pi'(X)f_X = \alpha(Y)S(f)\pi(X) - S'(f)\alpha(X)\pi(X) = (\alpha(Y)S(f) - S'(f)\alpha(X))\,\pi(X)$. In particular $\pi'(Y)g = \alpha(Y)S(f) - S'(f)\alpha(X)$.

Lemma 2.4. Let (B, S) be a Frobenius category and $f\colon X \to Y$ be a morphism in B. Then f is isomorphic to the residue class $\underline{u}$ of a proper monomorphism u.

Proof. Indeed, given a morphism $f\colon X \to Y$ of B and a proper monomorphism $X \xrightarrow{x} I(X)$ of X into an S-injective $I(X)$.

Consider canonical morphism $i\colon Y \to Y \oplus I(X)$ and $p\colon Y \oplus I(X) \to Y$. Then we have $pi = \mathrm{id}_Y$ and $\mathrm{id}_{Y \oplus I(X)} - ip = jq$, where $j\colon I(X) \to Y \oplus I(X)$ and $q\colon Y \oplus I(X) \to I(X)$ are canonical morphisms. So we have $\underline{\mathrm{id}_{Y \oplus I(X)}} = \underline{ip}$. It follows that $\underline{p}$ is an isomorphism in $\underline{\mathsf{B}}$. Now $p\left(\begin{smallmatrix} f \\ x \end{smallmatrix}\right) = f$, f is clearly isomorphic to the residue class of $\left(\begin{smallmatrix} f \\ x \end{smallmatrix}\right)$.

On the other hand S contains the short exact sequence

$$0 \longrightarrow X \xrightarrow{\left(\begin{smallmatrix} f \\ x \end{smallmatrix}\right)} Y \oplus I(X) \xrightarrow{(-g,y)} C \longrightarrow 0$$

where C is the pushout occurring in the following commutative diagram with rows in S.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "f"'] & I(X) \arrow[r, "\overline{x}"] \arrow[d, "y"'] & TX \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r, "g"] & C \arrow[r, "\overline{g}"] & TX \arrow[r] & 0 \end{tikzcd}$$

(Consider $0\colon Y \to TX$ and $\overline{x}\colon I(X) \to TX$, we have $\overline{x}x = 0 = 0f$. From the universal property there exists a unique $\overline{g}\colon C \to TX$ such that $\overline{g}g = 0$ and $\overline{g}y = \overline{x}$.)

It suffices to show that the sequence is exact.

(1) $\left(\begin{smallmatrix} f \\ x \end{smallmatrix}\right)$ is a monomorphism: Let $a\colon Z \to X$ satisfies $\left(\begin{smallmatrix} f \\ x \end{smallmatrix}\right) a = 0$, then $xa = 0$, so $a = 0$.

(2) $\ker(-g, y) = \operatorname{im}\left(\begin{smallmatrix} f \\ x \end{smallmatrix}\right)$: $(-g, y)\left(\begin{smallmatrix} f \\ x \end{smallmatrix}\right) = -gf + yx = 0$. Let $\left(\begin{smallmatrix} b \\ i \end{smallmatrix}\right) : Z \to Y \oplus I(X)$ with $(-g, y)\left(\begin{smallmatrix} b \\ i \end{smallmatrix}\right) = 0$, i.e., $gb = yi$, it suffices to show that there exists $a\colon Z \to X$ with $b = fa$ and $i = xa$. Since $\overline{g}g = 0$, $\overline{x}i = \overline{g}yi = \overline{g}gb = 0$. Thanks to the universal property, there exists a unique $a\colon Z \to X$ with $i = xa$. Since $gb = yi = yxa = gfa$, $b = fa$. Thus $\left(\begin{smallmatrix} b \\ i \end{smallmatrix}\right) = \left(\begin{smallmatrix} f \\ x \end{smallmatrix}\right) a$.

(3) $(-g, y)$ is an epimorphism: Let $c\colon Z \to C$, there exists $i\colon Z \to I(X)$ with $\overline{x}i = \overline{g}c$ since $\overline{x}$ is an epimorphism. Note that $\overline{g}(c - yi) = \overline{g}c - \overline{g}yi = \overline{g}c - \overline{x}i = 0$. So there exists a unique $b\colon Z \to Y$ with $gb = c - yi$. Therefore $(-g, y)\left(\begin{smallmatrix} -b \\ i \end{smallmatrix}\right) = gb + yi = c$.

$\square$

Remark 2.2. The bottom row of

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "f"'] & I(X) \arrow[r, "\overline{x}"] \arrow[d, "y"'] & TX \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r, "g"] & C \arrow[r, "\overline{g}"] & TX \arrow[r] & 0 \end{tikzcd}$$

is exact.

(1) $g\colon Y \to C$ is a monomorphism: From the property of pushout we have

$$X \xrightarrow{\left(\begin{smallmatrix} f \\ x \end{smallmatrix}\right)} Y \oplus I(X) \xrightarrow{(-g,y)} C \longrightarrow 0$$

is exact. Let $k\colon K \to Y$ satisfies $gk = 0$, it suffices to show that $k = 0$. Consider $\left(\begin{smallmatrix} k \\ 0 \end{smallmatrix}\right) : K \to Y \oplus I(X)$, we have $(-g, y)\left(\begin{smallmatrix} k \\ 0 \end{smallmatrix}\right) = 0$. From the exactness there exists $t\colon K \to X$ with $\left(\begin{smallmatrix} f \\ x \end{smallmatrix}\right) t = \left(\begin{smallmatrix} k \\ 0 \end{smallmatrix}\right)$, that is, $ft = k$ and $xt = 0$. Note that x is a monomorphism, we have $k = 0$.

(2) $\ker \overline{g} = \operatorname{im} g$: From the construction $\overline{g}g = 0$. Let $h\colon Z \to C$ with $\overline{g}h = 0$. From the exact sequence there exists $\left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right) : Z \to Y \oplus I(X)$ such that $(-g, y)\left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right) = -ge_1 + ye_2 = h$. It follows that $\overline{g}h = \overline{g}(-ge_1 + ye_2) = 0 + \overline{g}ye_2 = \overline{x}e_2$. Since $\overline{g}h = 0$, we have $\overline{x}e_2 = 0$. Since the top row is exact, there exists $s\colon Z \to X$ such that $xs = e_2$. Then $g(e_1 - fs) = ge_1 - gfs = ge_1 - yxs = ge_1 - ye_2 = h$. Thus h factors through g. Therefore $\ker \overline{g} = \operatorname{im} g$.

(3) $\overline{g}$ is an epimorphism: Since $\overline{x}$ is an epimorphism and $\overline{g}y = \overline{x}$, $\overline{g}$ is also an epimorphism.

Let C be an abelian category. An object $X \in \mathsf{C}$ is **simple** if it is not isomorphic to 0 and any subobject of X is either X or 0. We write $X \supset Y$ or $Y \subset X$ to denote a subobject Y of X. A sequence $X = X_0 \supset X_1 \supset \cdots \supset X_{n-1} \supset X_n = 0$ is a **composition series** if X_i/X_{i+1} is simple for all i with $0 \leq i < n$. Then the conditions below are equivalent:

(a) there exists a composition series $X = X_0 \supset X_1 \supset \cdots \supset X_{n-1} \supset X_n = 0$,

(b) there exists an integer n such that for any sequence $X = X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_m = 0$, we have $m \leq n$,

(c) any decreasing sequence $X = X_0 \supset X_1 \supset \cdots \supset X_m \supset \cdots$ is stationary (i.e. $X_m = X_{m+1}$ for $m \gg 0$) and any increasing sequence $X_0 \subset \cdots \subset X_m \subset \cdots \subset X$ is stationary.

One can verify the integer n in (a) depends only on X. If the equivalent conditions above are satisfied, we say that X has **finite length** and the integer n in (a) is called the **length** of X.

Let A be an abelian Frobenius category such that every object has finite length. This assumption ensures us that the classical Krull–Schmidt theorem is valid in A.

Lemma 2.5. Let X be in A indecomposable and not projective and $f\colon X \to Y$ an epimorphism for some $Y \in \mathsf{A}$. Then the residue class of f satisfies $\underline{f} \neq 0$ in $\underline{\mathsf{A}}$.

Proof. Suppose that $\underline{f} = 0$ in $\underline{\mathsf{A}}$. Then there exists a projective P in A and morphisms $g\colon X \to P$ and $h\colon P \to Y$ such that $f = gh$. Since P is projective and f is an epimorphism, there exists $g'\colon P \to X$ such that $h = fg'$. In particular $f = f(g'g)$. Assume conversely $g'g$ is an isomorphism, we have $\left((g'g)^{-1}g'\right)g = \mathrm{id}_X$, that is, g is split monomorphism. Thus X is a direct summand of P, which implies that X is projective, a contradiction. So the endomorphism $g'g$ of X is NOT an isomorphism. Note that in A, endomorphism ring of an indecomposable object is a local ring, so $\operatorname{End} X$ is a local ring. $g'g$ is not an isomorphism, $g'g \in \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of $\operatorname{End} X$. Since $\operatorname{End} X$ is artinian, $\mathfrak{m}$ is nilpotent. It follows that $g'g$ is nilpotent. Since $f = f(g'g)$, $f = f(g'g)^n = 0$, a contradiction. $\square$

Corollary 2.1. Let X, Y be in A indecomposable and not projective. If $\operatorname{Hom}(X,Y) \neq 0$, then there exists $Z \in \mathsf{A}$ such that $\underline{\operatorname{Hom}}(X,Z) \neq 0 \neq \underline{\operatorname{Hom}}(Z,Y)$.

Proof. Let $0 \neq f \in \operatorname{Hom}(X,Y)$. Consider a factorization of $f = gh$ such that g is a monomorphism and h is an epimorphism. Let $Z = \operatorname{Im}(h)$. By the Lemma 2.5 and its dual we infer that g and h yield non-zero morphisms in $\underline{\mathsf{A}}$. $\square$

Let (B, S) be a Frobenius category and $\underline{\mathsf{B}}$ the stable category. We define a set T of sextuples in $\underline{\mathsf{B}}$. Let $X, Y \in \mathsf{B}$ and $u \in \operatorname{Hom}_{\mathsf{B}}(X,Y)$. Consider the following diagram in B:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "x"'] & Y \arrow[d, "v"] \\
I(X) \arrow[r, "\overline{u}"] \arrow[d, "\overline{x}"'] & C_u \arrow[d, "w"] \\
TX \arrow[r, equal] & TX
\end{tikzcd}$$

where $0 \to X \xrightarrow{x} I(X) \xrightarrow{\overline{x}} TX \to 0$ is in S and $I(X)$ is S-injective. C_u is the pushout of u and x.

Since B is closed under extensions in some abelian category A the pushout C_u in B coincides with the pushout in A. A sextuple of the form $X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} TX$ and its image in $\underline{\mathsf{B}}$ will be called **standard**.

A sextuple $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ of objects and morphisms in $\underline{\mathsf{B}}$ belongs to T if it is isomorphic in $\underline{\mathsf{B}}$ to a standard sextuple.

Theorem 2.1. The set T is a triangulation of $\underline{\mathsf{B}}$.

Proof. To begin with we prove a Lemma.

Lemma 2.6. Let (B, S) be a Frobenius category. Assume that $0 \to Y \xrightarrow{y} I(Y) \xrightarrow{\overline{y}} TY \to 0$ and $0 \to Y \xrightarrow{m} I \xrightarrow{p} M \to 0$ are exact sequences in S, where $I(Y)$ and I are S-injectives. Suppose we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "y"] \arrow[d, "v"'] & I(Y) \arrow[r, "\overline{y}"] \arrow[d, "\overline{v}"'] & TY \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r, "j"] & X' = C_v \arrow[r, "j'"] & TY \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "m"] \arrow[d, "v"'] & I \arrow[r, "p"] \arrow[d, "\overline{v}'"'] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r, "\tilde{j}"] & \tilde{X}' = C_v \arrow[r, "\tilde{j}'"] & TY \arrow[r] & 0
\end{tikzcd}$$

Then

(1) There exists $\alpha, \alpha', \beta, \beta'$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "y"] \arrow[d, equal] & I(Y) \arrow[r, "\overline{y}"] \arrow[d, "\alpha"'] & TY \arrow[r] \arrow[d, "\beta"'] & 0 \\
0 \arrow[r] & Y \arrow[r, "m"] \arrow[d, equal] & I \arrow[r, "p"] \arrow[d, "\alpha'"'] & M \arrow[r] \arrow[d, "\beta'"'] & 0 \\
0 \arrow[r] & Y \arrow[r, "y"] & I(Y) \arrow[r, "\overline{y}"] & TY \arrow[r] & 0
\end{tikzcd} \tag{1}$$

Moreover, for arbitrary $\alpha, \alpha', \beta, \beta'$ makes (1) commutes, there exists $r\colon X' \to \tilde{X}'$ and $r'\colon \tilde{X}' \to X'$ such that

$$\begin{aligned}
&\underline{r'r} = \underline{\mathrm{id}_{X'}}, && \underline{rr'} = \underline{\mathrm{id}_{\tilde{X}'}}, \\
&r\overline{v} = \overline{v}'\alpha, && rj = \tilde{j}, \\
&r'\overline{v}' = \overline{v}\alpha', && r'\tilde{j} = j, \\
&\tilde{j}'r = \beta j', && j'r' = \beta'\tilde{j}';
\end{aligned}$$

(2) There exists isomorphism of triangles

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "j"] \arrow[d, equal] & X' = C_v \arrow[r, "j'"] \arrow[d, dashed, "r"] & TY \arrow[d, equal] \\
Y \arrow[r, "v"] & Z \arrow[r, "\tilde{j}"] & \tilde{X}' = C'_v \arrow[r, "\beta'\tilde{j}'"] & TY
\end{tikzcd}$$

Proof. From the proof of Lemma 2.4 we obtain (1). Note that $(\alpha'\alpha - \mathrm{id}_{I(Y)})y = \alpha'\alpha y - y = 0$, $\alpha'\alpha - \mathrm{id}_{I(Y)}$ factors through $\overline{y}$ thanks to the universal property. Thus there exists $a\colon TY \to I(Y)$ such that

$$\alpha'\alpha - \mathrm{id}_{I(Y)} = a\overline{y}, \quad \beta'\beta - \mathrm{id}_{TY} = \overline{y}a,$$

and $\underline{\beta'\beta} = \underline{\mathrm{id}_{TY}}$, $\underline{\beta\beta'} = \underline{\mathrm{id}_M}$. Consider the following pushout square

$$\begin{tikzcd}
Y \arrow[r, "y"] \arrow[d, "v"'] & I(Y) \arrow[d, "\overline{v}"] \\
Z \arrow[r, "j"] & X'
\end{tikzcd}$$

and morphism $\left(\tilde{j}, \overline{v}'\alpha\right)$, where $\tilde{j}\colon Z \to \tilde{X}'$ and $\overline{v}'\alpha\colon I(Y) \to I \to \tilde{X}'$. Since $\tilde{j}v = \overline{v}'m = \overline{v}'\alpha\overline{y}$, there exists a morphism $r\colon X' \to \tilde{X}'$ such that

$$r\overline{v} = \overline{v}'\alpha, \quad rj = \tilde{j}.$$

Similarly there exists $r' \colon \tilde{X}' \to X'$ such that

$$r'\overline{v}' = \overline{v}\alpha', \quad r'\tilde{j} = j.$$

This leads to $r'rj = j$, so $(rr' - \mathrm{id}_{X'})j = 0$, and so we have morphism $b\colon TY \to X'$ such that $r'r - \mathrm{id}_{X'} = bj'$. It follows that

$$(r'r - \mathrm{id}_{X'})\overline{v} = bj'\overline{v} = b\overline{y}.$$

On the other hand

$$(r'r - \mathrm{id}_{X'})\overline{v} = \overline{v}(\alpha'\alpha - \mathrm{id}_{I(Y)}) = \overline{v}a\overline{y}.$$

So $b = \overline{v}a$. Then $\underline{b} = 0$ and $\underline{r'r} = \underline{\mathrm{id}_{X'}}$. Similarly $\underline{rr'} = \underline{\mathrm{id}_{\tilde{X}'}}$.

Consider epimorphism $(j, -\overline{v}) : Z \oplus I(Y) \to X'$. Since

$$\tilde{j}'r\,(j, -\overline{v}) = \left(\tilde{j}'\tilde{j}, -\tilde{j}'\overline{v}'\alpha\right) = (0, -p\alpha) = (0, -\beta\overline{y}) = (\beta j'j, -\beta j'\overline{v}) = \beta j'\,(j, -\overline{v})\,,$$

So $\tilde{j}'r = \beta j'$. Similarly $j'r' = \beta'\tilde{j}'$. $\square$

Now we check the axioms from Definition 1.4.

(TR1) By definition T is closed under isomorphisms and every morphism can be embedded into a triangle. Clearly the sextuple $X \xrightarrow{\mathrm{id}_X} X \xrightarrow{0} 0 \xrightarrow{0} TX$ belongs to T.

(TR2) It is easily seen that it suffices to consider the case of a standard triangle. Suppose that $X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} TX$ is a standard triangle. Let $0 \to Y \xrightarrow{y} I(Y) \xrightarrow{\overline{y}} TY \to 0$ be in S with $I(Y)$ being S-injective. There exists I_u such that $I_u x = yu$ and there exists Tu such that $\overline{y}I_u = (Tu)\overline{x}$.

Define $f\colon C_u \to I(Y)$ by using the pushout property:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "x"'] & Y \arrow[d, "v"] \arrow[ddr, "y"] & \\
I(X) \arrow[r, "\overline{u}"] \arrow[drr, "I_u"'] & C_u \arrow[dr, dashed] & \\
& & I(Y)
\end{tikzcd}$$

thus $y = fv$, $I_u = f\overline{u}$. Since $\overline{y}fv = \overline{y}y = 0 = (Tu)wv$, and $\overline{y}f\overline{u} = \overline{y}I_u = \overline{x}Tu = (Tu)w\overline{u}$, we infer that $\overline{y}f = (Tu)w$, for C_u is a pushout.

In this way, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & Y \arrow[r, "v"] \arrow[d, "y"'] & C_u \arrow[r, "w"] \arrow[d, "\binom{f}{w}"'] & TX \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I(Y) \arrow[r, "\binom{\mathrm{id}_{I(Y)}}{0}"] \arrow[d, "\overline{y}"'] & I(Y)\oplus TX \arrow[r, "(0,\mathrm{id}_{TX})"] \arrow[d, "(\overline{y},-Tu)"] & TX \arrow[r] & 0 \\
 & TY \arrow[r, equal] \arrow[d] & TY \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

The upper two rows are exact. We claim that the left upper square is a pushout, and it suffices to show the square satisfies the universal property. For arbitrary Z and $\varphi\colon C_u \to Z$ and $\psi\colon I(Y) \to Z$ with $\varphi v = \psi y$. Let $\theta_1 = \psi$. Since $w\colon C_u \to TX$ is an epimorphism, there exists a unique θ_2 such that $\theta_2 w = \varphi - \psi f$. Clearly $\theta = (\theta_1, \theta_2)$ is the unique morphism from $I(Y) \oplus TX$ to Z satisfies $\theta \binom{f}{w} = \varphi$ and $\theta \binom{\mathrm{id}_{I(Y)}}{0} = \psi$. Therefore,

$$Y \xrightarrow{v} C_u \xrightarrow{\binom{f}{w}} I(Y)\oplus TX \xrightarrow{(\overline{y},-Tu)} TY$$

is a standard sextuple. Now consider

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, equal] & C_u \arrow[r, "\binom{f}{w}"] \arrow[d, equal] & I(Y)\oplus TX \arrow[r, "(\overline{y},-Tu)"] \arrow[d, "\underline{\pi_2}"] & TY \arrow[d, equal] \\
Y \arrow[r, "\underline{v}"] & C_u \arrow[r, "\underline{w}"] & TX \arrow[r, "\underline{-Tu}"] & TY
\end{tikzcd}$$

Clearly the left and middle square are commutative in B. Since $(\overline{y},-Tu)-(-Tu)\pi_2 = (\overline{y},0)$ and $(\overline{y},0) : I(Y)\oplus TX \to TY$ can be factored as $I(Y)\oplus TX \xrightarrow{\pi_1} I(Y) \xrightarrow{\overline{y}} TY$, we have $\underline{(\overline{y},0)} = 0$ in $\underline{\mathsf{B}}$. Finally we show that $\underline{\pi_2}$ is an isomorphism. Define $\iota_2\colon TX \to I(Y)\oplus TX$ be canonical embedding. Then $\pi_2\iota_2 = \mathrm{id}_{TX}$. Since $\mathrm{id}_{I(Y)\oplus TX} - \iota_2\pi_2 = \begin{pmatrix} \mathrm{id}_{I(Y)} \\ 0 \end{pmatrix}$, and $\begin{pmatrix} \mathrm{id}_{I(Y)} \\ 0 \end{pmatrix}$ can be factored as $I(Y)\oplus TX \xrightarrow{\pi_1} I(Y) \xrightarrow{\iota_1} I(Y)\oplus TX$, we obtain $\underline{\iota_2\pi_2} = \underline{\mathrm{id}_{I(Y)\oplus TX}}$. Consequently

$$Y \xrightarrow{v} C_u \xrightarrow{\binom{f}{w}} I(Y)\oplus TX \xrightarrow{(\overline{y},-Tu)} TY$$

is isomorphic to $Y \xrightarrow{\underline{v}} C_u \xrightarrow{\underline{w}} TX \xrightarrow{\underline{-Tu}} TY$ in $\underline{\mathsf{B}}$.

(TR3) Let us first consider the case of standard triangles. Consider the following two standard sextuples:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "x"'] & Y \arrow[d, "v"] \\ I(X) \arrow[r, "\overline{u}"] \arrow[d, "\overline{x}"'] & C_u \arrow[d, "w"'] \\ TX \arrow[r, equal] & TX \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X' \arrow[r, "u'"] \arrow[d, "x'"'] & Y' \arrow[d, "v'"] \\ I(X') \arrow[r, "\overline{u'}"] \arrow[d, "\overline{x'}"'] & C_{u'} \arrow[d, "w'"'] \\ TX' \arrow[r, equal] & TX' \end{tikzcd}$$

and two morphisms f and g such that $\underline{u'f} = \underline{gu}$ in $\underline{\mathsf{B}}$. Then there exists $\alpha\colon I(X) \to Y'$ such that $gu = u'f + \alpha x$. We have morphisms $I_f\colon I(X) \to I(X')$ such that $x'f = I_f x$ and $Tf\colon TX \to TX'$ such that $\overline{x'}I_f = Tf\overline{x}$. So we obtain morphisms $v'g\colon Y \to C_{u'}$, and $\overline{u'}I_f + v'\alpha\colon I(X) \to C_{u'}$, such that $v'gu = v'u'f + v'\alpha x = \overline{u'}x'f + v'\alpha x = \left(I_f\overline{u'} + v'\alpha\right)x$. This yields a morphism $h\colon C_u \to C_{u'}$, such that $hv = v'g$ and $h\overline{u} = \overline{u'}I_f + v'\alpha$, for C_u is a pushout. We claim that $w'h = Tfw$. For this it is enough to show that $w'hv = Tfwv$ and $w'h\overline{u} = Tfw\overline{u}$ thanks to the universal property of pushout. For the first observe that $Tfwv = 0$ and $w'hv = w'v'g = 0$ For the second we have $Tfw\overline{u} = Tf\overline{x} = \overline{x'}I_f = w\overline{u'}I_f = w'\overline{u'}I_f + w'v'\alpha = w'\left(\overline{u'}I_f + v'\alpha\right) = w'h\overline{u}$.

Thus $(\underline{f}, \underline{g}, \underline{h})$ is a morphism of triangles.

The general case is easily deduced from the previous one. In fact, let $(X, Y, Z, \underline{u}, \underline{v}, \underline{w})$ and $(X', Y', Z', \underline{u'}, \underline{v'}, \underline{w'})$ be arbitrary elements in T and f, g two morphisms such that $\underline{u'f} = \underline{gu}$ in $\underline{\mathsf{B}}$. Then we have isomorphisms to the corresponding standard triangles. Using the first part we obtain the following commutative diagram, where the rows are in T and $\underline{h_1}, \underline{h_2}$ are isomorphisms:

$$\begin{tikzcd} X \arrow[r, "\underline{u}"] \arrow[d, equal] & Y \arrow[r, "\underline{v}"] \arrow[d, equal] & Z \arrow[r, "\underline{w}"] \arrow[d, "\underline{h_1}"] & TX \arrow[d, equal] \\ X \arrow[r, "\underline{u}"] \arrow[d, "\underline{f}"] & Y \arrow[r, "\underline{\tilde{v}}"] \arrow[d, "\underline{g}"] & C_u \arrow[r, "\underline{\tilde{w}}"] \arrow[d, "\underline{h}"] & TX \arrow[d, "\underline{Tf}"] \\ X' \arrow[r, "\underline{u'}"] \arrow[d, equal] & Y' \arrow[r, "\underline{\tilde{v}'}"] \arrow[d, equal] & C_{u'} \arrow[r, "\underline{\tilde{w}'}"] & TX' \arrow[d, equal] \\ X' \arrow[r, "\underline{u'}"] & Y' \arrow[r, "\underline{v'}"] & Z' \arrow[r, "\underline{w'}"] \arrow[u, "\underline{h_2}"'] & TX' \end{tikzcd}$$

Then $\left(\underline{f}, \underline{g}, \underline{h_2^{-1}hh_1}\right)$ is a morphism of triangles. In fact, $\underline{h_2^{-1}hh_1v} = \underline{h_2^{-1}h\tilde{v}} = \underline{h_2^{-1}}\underline{\tilde{v}'g} = \underline{v'g}$ and $\underline{w'h_2^{-1}hh_1} = \underline{\tilde{w}'}\underline{hh_1} = \underline{Tf}\underline{\tilde{w}}\underline{h_1} = \underline{Tfw}$.

(TR4) Consider the diagram in $\underline{\mathsf{B}}$:

$$\begin{tikzcd}
X \arrow[r, "\underline{u}"] \arrow[d, equal] & Y \arrow[r, "\underline{i}"] \arrow[d, "\underline{v}"] & Z' \arrow[r, "\underline{i'}"] \arrow[d, dashed, "\underline{f}"] & TX \arrow[d, equal] \\
X \arrow[r, "\underline{vu}"] & Z \arrow[r, "\underline{k}"] \arrow[d, "\underline{j}"] & Y' \arrow[r, "\underline{k'}"] \arrow[d, dashed, "\underline{g}"] & TX \arrow[d, "\underline{Tu}"] \\
& X' \arrow[r, equal] \arrow[d, "\underline{j'}"] & X' \arrow[r, "\underline{j'}"] \arrow[d, "(\underline{Ti})\underline{j'}"] & TY \\
& TY \arrow[r, "\underline{Ti}"] & TZ' &
\end{tikzcd} \tag{2}$$

where the first row, the second row, and the second column are all standard triangles. We need to prove that there exist morphisms $f\colon Z' \to Y'$ and $g'\colon Y' \to X'$ such that the triangle in the third column of (2) is isomorphic to a standard triangle, and every square in (2) is commutative.

Step 1. Since the first row is a standard triangle, we obtain the following commutative diagram induced by the pushout.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "u"'] & I(X) \arrow[r, "\overline{x}"] \arrow[d, "\overline{u}"'] & TX \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "i"] & Z' = C_u \arrow[r, "i'"] & TX \arrow[r] & 0
\end{tikzcd} \tag{3}$$

Since the second column is a standard triangle, we obtain the following commutative diagram induced by the pushout.

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "y"] \arrow[d, "v"'] & I(Y) \arrow[r, "\overline{y}"] \arrow[d, "\overline{v}"'] & TY \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r, "j"] & X' = C_v \arrow[r, "j'"] & TY \arrow[r] & 0
\end{tikzcd} \tag{4}$$

Since the second row is a standard triangle, we obtain the following commutative diagram induced by the pushout.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "vu=w"'] & I(X) \arrow[r, "\overline{x}"] \arrow[d, "\overline{w}"'] & TX \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r, "k"] & Y' = C_w \arrow[r, "k'"] & TX \arrow[r] & 0
\end{tikzcd} \tag{5}$$

Consider the pushout square

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "u"'] & I(X) \arrow[d, "\overline{u}"] \\
Y \arrow[r, "i"] & Z'
\end{tikzcd}$$

and $(kv, \overline{w})$, where $kv\colon Y \to Z \to Y'$ and $\overline{w}\colon I(X) \to Y'$. Since $(kv)u = \overline{w}x$, there exists $f\colon Z' \to Y'$ with

$$fi = kv, \quad f\overline{u} = \overline{w}. \tag{6}$$

Since $(i, -\overline{u}) : Y \oplus I(X) \to Z'$ is an epimorphism and

$$k'f\,(i, -\overline{u}) \overset{(6)}{=} (k'kv, -k'\overline{w}) \overset{(5)}{=} (0, -\overline{x}) \overset{(3)}{=} i'\,(i, -\overline{u})\,,$$

we obtain

$$k'f = i'. \tag{7}$$

Step 2. In order to construct $g\colon Y' \to X'$, we need to make some preparations. Consider the morphism $z'i\colon Y \to Z \to I(Z')$. Let $M = \mathrm{Coker}(z'i)$, we obtain an exact sequence

$$0 \longrightarrow Y \xrightarrow{z'i} I(Z') \xrightarrow{\pi} M \longrightarrow 0$$

Note that $(\pi z')i = \pi(z'i) = 0$, there exists $t\colon TX \to M$ such that the two squares in the upper part of the diagram are commutative.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, equal] & Z' \arrow[r, "i'"] \arrow[d, "z'"'] & TX \arrow[r] \arrow[d, "t"] & 0 \\
0 \arrow[r] & Y \arrow[r, "z'i"] & I(Z') \arrow[r, "\pi"] \arrow[d, "\overline{z'}"'] & M \arrow[r] \arrow[d] & 0 \\
 & & TZ' \arrow[r, equal] \arrow[d] & TZ' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{8}$$

Since π is surjective, the upper right square in (8) is both the pullback square of (t, π) and the pushout square of (z', i'). Therefore, $\mathrm{Coker}\, t = \mathrm{Coker}(z') = TZ'$, and t is injective. Moreover, since each square in (8) is commutative and B is a full subcategory closed under extensions in A it follows that $M \in \mathsf{B}$. Thus, we obtain the following two exact sequences in S,

$$0 \longrightarrow Y \xrightarrow{y} I(Y) \xrightarrow{\overline{y}} TY \longrightarrow 0$$

and

$$0 \longrightarrow Y \xrightarrow{z'i} I(Z') \xrightarrow{\pi} M \longrightarrow 0$$

where $I(Y)$ and $I(Z')$ are both S-injective. So we have commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "y"] \arrow[d, equal] & I(Y) \arrow[r, "\overline{y}"] \arrow[d, "\alpha"'] & TY \arrow[r] \arrow[d, "\beta"'] & 0 \\
0 \arrow[r] & Y \arrow[r, "z'i"] \arrow[d, equal] & I(Z') \arrow[r, "\pi"] \arrow[d, "\alpha'"'] & M \arrow[r] \arrow[d, "\beta'"'] & 0 \\
0 \arrow[r] & Y \arrow[r, "y"] & I(Y) \arrow[r, "\overline{y}"] & TY \arrow[r] & 0
\end{tikzcd} \tag{9}$$

Since we already have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "y"] \arrow[d, "i"'] & I(Y) \arrow[r, "\overline{y}"] \arrow[d, "I(i)"'] & TY \arrow[r] \arrow[d, "Ti"'] & 0 \\
0 \arrow[r] & Z' \arrow[r, "z'"] & I(Z') \arrow[r, "\overline{z'}"] & TZ' \arrow[r] & 0
\end{tikzcd} \tag{10}$$

We can choose $\alpha = I(i)$ (we emphasize that this is a crucial choice). From Lemma 2.3 we have $\underline{\beta'\beta} = \underline{\mathrm{id}_{TY}}$ and $\underline{\beta\beta'} = \underline{\mathrm{id}_M}$.

Using the exact sequence $0 \to Y \xrightarrow{z'i} I(Z') \xrightarrow{\pi} M \to 0$ in S, we take the pushout of $(z'i, v)$ and obtain the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "z'i"] \arrow[d, "v"'] & I(Z') \arrow[r, "\pi"] \arrow[d, "\overline{v}'"'] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r, "\tilde{j}"] & \tilde{X}' = C'_v \arrow[r, "\tilde{j}'"] & M \arrow[r] & 0
\end{tikzcd} \tag{11}$$

From Lemma 2.6 (1), there exists $r\colon X' \to \tilde{X}'$ and $r'\colon \tilde{X}' \to X'$ such that

$$\begin{aligned}
&\underline{r'r} = \underline{\mathrm{id}_{X'}}, \quad \underline{rr'} = \underline{\mathrm{id}_{\tilde{X}'}} \\
&r\overline{v} = \overline{v}'\alpha, \qquad rj = \tilde{j} \\
&r'\overline{v}' = \overline{v}\alpha', \quad r'\tilde{j} = j.
\end{aligned} \tag{12}$$

Consider pushout square

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "vu"'] & I(X) \arrow[d, "\overline{w}"] \\
Z \arrow[r, "k"] & Y'
\end{tikzcd}$$

and $(\tilde{j}, \overline{v}'z'\overline{u})$, here $\tilde{j}\colon Z \to \tilde{X}'$ and $\overline{v}'z'\overline{u}\colon I(X) \to Z' \to I(Z') \to X'$. Since

$$\tilde{j}(vu) \overset{(11)}{=} \overline{v}'z'iu \overset{(3)}{=} \overline{v}'z'\overline{ux},$$

there exists $g'\colon Y' \to \tilde{X}'$ such that

$$g'k = \tilde{j}, \quad g'\overline{w} = \overline{v}'z'\overline{u} \tag{13}$$

thanks to the universal property.

Step 3. We claim that the following square

$$\begin{array}{ccc} Z' & \xrightarrow{z'} & I(Z') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{v}'} \\ Y' & \xrightarrow{g'} & \tilde{X}' \end{array}$$

In fact, since $(i, -\overline{u}) : Y \oplus I(X) \to Z'$ is an epimorphism, and

$$g'f\,(i, -\overline{u}) \overset{(6)}{=} (g'kv, -g'\overline{w}) \overset{(13)}{=} \left(\tilde{j}v, -\overline{v}'z'\overline{u}\right) \overset{(11)}{=} (\overline{v}z'i, -\overline{v}'z'\overline{u}) = \overline{v}'z'\,(i, -\overline{u})$$

we have $g'f = \overline{v}'z'$. Assume that (a, b) satisfies $af = bz'$, where $a\colon Y' \to D$ and $b\colon I(Z') \to D$. Since

$$\begin{array}{ccc} Y & \xrightarrow{zi} & I(Z') \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle \overline{v}'} \\ Z & \xrightarrow{\tilde{j}} & \tilde{X}' \end{array}$$

is pushout, and $ak\colon Z \to Y' \to D$, $b\colon I(Z') \to D$ satisfy

$$bz'i = afi \overset{(6)}{=} akv,$$

there exists $s\colon \tilde{X}' \to D$ such that

$$s\tilde{j} = ak, \quad s\overline{v}' = b$$

thanks to the universal property. Now we prove $sg' = a$. Consider epimorphism $(k, -\overline{w}) : Z \oplus I(X) \to Y'$. Since

$$sg'\,(k, -\overline{w}) \overset{(13)}{=} \left(s\tilde{j}, -s\overline{v}'z'\overline{u}\right) = (ak, -bz'\overline{u}) = (ak, -af\overline{u}) \overset{(6)}{=} a\,(k, -\overline{w}),$$

we obtain $sg' = a$. Finally we show such s is unique. If $t\colon \tilde{X}' \to D$ satisfies $tg' = a$ and $t\overline{v}' = b$, then $t\tilde{j} = tg'k = ak$. Therefore from the definition of pushout, $t = s$.

Therefore we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z' & \xrightarrow{z'} & I(Z') & \xrightarrow{\overline{z'}} & TZ' & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle \overline{v}'}\downarrow & & \| & & \\ 0 & \longrightarrow & Y' & \xrightarrow{g'} & \tilde{X}' & \xrightarrow{w} & TZ' & \longrightarrow & 0 \end{array} \tag{14}$$

and triangle

$$Z' \xrightarrow{\underline{f}} Y' \xrightarrow{\underline{g'}} \tilde{X}' \xrightarrow{\underline{w}} TZ'$$

in $\underline{\mathsf{B}}$. Here $\underline{w}$ will be explained in the next step.

Now let $g = r'g' \colon Y' \to \tilde{X}' \to X'$. From (13) and (12) we have

$$gk = j, \quad g\overline{w} = \overline{v}\alpha' z' \overline{u}. \tag{15}$$

Since we have isomorphisms of triangles in $\underline{\mathsf{B}}$

$$\begin{tikzcd}
Z' \arrow[r, "f"] \arrow[d, equal] & Y' \arrow[r, "g'"] \arrow[d, equal] & \tilde{X}' \arrow[r, "w"] \arrow[d, "r'"'] & TZ' \arrow[d, equal] \\
Z' \arrow[r, "f"] & Y' \arrow[r, "g"] & X' \arrow[r, "wr"] & TZ'
\end{tikzcd}$$

Thus

$$Z' \xrightarrow{f} Y' \xrightarrow{g'} X' \xrightarrow{wr} TZ'$$

belongs to T.

Step 4. We claim that

$$wr = (Ti)j'. \tag{16}$$

In fact, since $(j, -\overline{v}) : Z \oplus I(Y) \to X'$ is an epimorphism, and

$$wr\,(j, -\overline{v}) \overset{(12)}{=} \left(w\tilde{j}, -w\overline{v}' I(i)\right) \overset{(13)}{=} (wg'k, -w\overline{v}' I(i)) \overset{(14)}{=} (0, -\overline{z}' I(i)) \overset{(10)}{=} (0, -(Ti)\overline{y}) \overset{(4)}{=} (Ti)j'\,(j, -\overline{v}),$$

the assertion holds, and hence $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{(Ti)j'} TZ'$ belongs to T.

Step 5. Now we proceed to verify all the commutative relations in (2). According to (6), (7), and (15), it suffices to verify

$$\underline{j'g} = \underline{(Tu)k'}.$$

From (3), (8) and (9) we obtain the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "u"'] & I(X) \arrow[r, "\overline{x}"] \arrow[d, "\overline{u}"'] & TX \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, equal] & Z' \arrow[r, "i'"] \arrow[d, "z'"'] & TX \arrow[r] \arrow[d, "t"'] & 0 \\
0 \arrow[r] & Y \arrow[r, "z'i"] \arrow[d, equal] & I(Z') \arrow[r, "\pi"] \arrow[d, "\alpha'"'] & M \arrow[r] \arrow[d, "\beta'"'] & 0 \\
0 \arrow[r] & Y \arrow[r, "y"] & I(Y) \arrow[r, "\overline{y}"] & TY \arrow[r] & 0
\end{tikzcd}$$

Then we obtian commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "u"'] & I(X) \arrow[r, "\overline{x}"] \arrow[d, "\alpha' z' \overline{u}"'] & TX \arrow[r] \arrow[d, "\beta' t"'] & 0 \\
0 \arrow[r] & Y \arrow[r, "y"] & I(Y) \arrow[r, "\overline{y}"] & TY \arrow[r] & 0
\end{tikzcd}$$

On the other hand we have commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "x"] \arrow[d, "u"'] & I(X) \arrow[r, "\overline{x}"] \arrow[d, "I(u)"'] & TX \arrow[r] \arrow[d, "Tu"'] & 0 \\
0 \arrow[r] & Y \arrow[r, "y"] & I(Y) \arrow[r, "\overline{y}"] & TY \arrow[r] & 0
\end{tikzcd}$$

Therefore $(\alpha' z' \overline{u} - I(u))\, x = 0$, so there exists $a\colon TX \to I(Y)$ such that

$$\alpha' z' \overline{u} - I(u) = a\overline{x}. \tag{17}$$

Consider epimorphism $(k, -\overline{w}) : Z \oplus I(X) \to Y'$. We have

$$j' g\,(k, -\overline{w}) \overset{(15)}{=} j'\,(j, -\overline{v}\alpha' z' \overline{u}) \overset{(4)}{=} (0, -\overline{y}\alpha' z' \overline{u})\,,$$

and

$$(Tu)k'\,(k, -\overline{w}) \overset{(5)}{=} (0, -(Tu)\overline{x}) = (0, -\overline{y}I(u))\,,$$

so

$$(j'g - (Tu)k')\,(k, -\overline{w}) = (0, -\overline{y}\,(\alpha' z' \overline{u} - I(u))) \overset{(17)}{=} (0, -\overline{y}a\overline{x}) \overset{(5)}{=} \overline{y}ak'\,(k, -\overline{w})\,,$$

and so

$$j'g - Tuk' = \overline{y}ak'.$$

It follows that

$$\underline{j'g} = \underline{(Tu)k'}.$$

This finishes the proof of the theorem. $\square$

Lemma 2.7. Let (B, S) be a Frobenius category. Then the elements in S give rise to triangles in $\underline{\mathsf{B}}$.

Say precisely, let $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is short exact sequence in S. Then $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} TX$ is a triangle in $\underline{\mathsf{B}}$, where w is a morphism in B makes the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, dashed, "\sigma"] & Z \arrow[r] \arrow[d, dashed, "w"] & 0 \\
0 \arrow[r] & X \arrow[r, "x"] & I(X) \arrow[r, "\overline{x}"] & TX \arrow[r] & 0
\end{tikzcd} \tag{18}$$

Moreover, any two such morphisms w and w' that make the diagram commute will yield isomorphic triangles.

Conversely, suppose $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{-w'} TX'$ is a triangle in $\underline{\mathsf{B}}$, Then, there exists a short exact sequence $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ in S, such that the induced triangle $X' \xrightarrow{\underline{u'}} Y' \xrightarrow{\underline{v'}} Z' \xrightarrow{\underline{-w'}}$

TX' is isomorphic to the given distinguished triangle, where w is the morphism in B that makes (18) commute.

Proof. To begin with we show the existence of σ and w in (18). Since $I(X)$ is S-injective, there exists σ with $\sigma u = x$. Now $(\overline{x}\sigma)\, u = \overline{x}(\sigma u) = \overline{x}x = 0$, there exists w with $wv = \overline{x}\sigma$ thanks to the universal property.

Let $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is a short exact sequence in S. Then we have the following square with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, "x"'] & Y \arrow[r, "v"] \arrow[d, "\left(\begin{smallmatrix}\sigma\\ v\end{smallmatrix}\right)"'] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I(X) \arrow[r, "\left(\begin{smallmatrix}\mathrm{id}_{I(X)}\\ 0\end{smallmatrix}\right)"] \arrow[d, "\overline{x}"'] & I(X)\oplus Z \arrow[r, "(0,\mathrm{id}_Z)"] \arrow[d, "(\overline{x},-w)"'] & Z \arrow[r] & 0 \\
 & TX \arrow[r, equal] \arrow[d] & TX \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

We now check the exactness of the second column.

(1) $\left(\begin{smallmatrix}\sigma\\ v\end{smallmatrix}\right)$ is an monomorphism: Assume that $\left(\begin{smallmatrix}\sigma\\ v\end{smallmatrix}\right) f = 0$, we have $vf = 0$. Since $\ker v = \operatorname{im} u$, there exists g such that $ug = f$. It follows that $\sigma ug = 0$, so $xg = 0$, and so $g = 0$ because x is an monomorphism. Thus $f = 0$.

(2) $\ker(\overline{x}, -w) = \operatorname{im}\left(\begin{smallmatrix}\sigma\\ v\end{smallmatrix}\right)$: On the one hand $(\overline{x}, -w)\left(\begin{smallmatrix}\sigma\\ v\end{smallmatrix}\right) = \overline{x}\sigma - wv = 0$. On the other hand, if $(\overline{x}, -w)\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)$, we have $\overline{x}\alpha_1 = w\alpha_2$. It suffices to show that Y is pullback of Z and $I(X)$ along w and $\overline{x}$. Since v is an epimorphism, there exists $\gamma\colon A \to Y$ with $v\gamma = \alpha_2$. Then $\overline{x}(\alpha_1 - \sigma\gamma) - = \overline{x}\alpha_1 - \overline{x}\sigma\gamma = w\alpha_2 - wv\gamma = w\alpha_2 - w\alpha_2 = 0$. From the universal property of kernel, there exists a unique $\delta\colon A \to X$ with $x\delta = \alpha_1 - \sigma\gamma$. Let $\beta = \gamma + u\delta$, we have $v\beta = v\gamma + vu\delta = \alpha_2$ and $\sigma\beta = \sigma\gamma + \sigma u\delta = \sigma\gamma + x\delta = \alpha_1$. If β_1, β_2 satisfy $\sigma\beta_i = \alpha_1$ and $v\beta_i = \alpha_2$, we have $\left(\begin{smallmatrix}\sigma\\ v\end{smallmatrix}\right)(\beta_1 - \beta_2) = 0$. Thus $\beta_1 = \beta_2$ from (1).

(3) $(\overline{x}, -w)$ is an epimorphism: Clearly from the fact that $\overline{x}$ is an epimorphism.

Thus from the proof of (TR2) in Theorem 2.1, the left upper square is a pushout. Consequently $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} TX$ is a triangle in $\underline{\mathsf{B}}$.

For the converse if $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{-w'} TX'$ is a triangle in $\underline{\mathsf{B}}$, then it is isomorphic to standard triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} TX$. From the proof of Lemma 2.4 we know that there exists a short exact sequence

$$0 \longrightarrow X \xrightarrow{\binom{u}{x}} Y \oplus I(X) \xrightarrow{(v,-\overline{u})} C_u \longrightarrow 0$$

in S. Moreover there exists w makes the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\binom{u}{x}"] \arrow[d, equal] & Y \oplus I(X) \arrow[r, "(v,-\overline{u})"] \arrow[d, "(0,\mathrm{id}_{I(X)})"] & C_u \arrow[r] \arrow[d, "-w"] & 0 \\
0 \arrow[r] & X \arrow[r, "x"] & I(X) \arrow[r, "\overline{x}"] & TX \arrow[r] & 0
\end{tikzcd}$$

This implies that the standard triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} TX$ is precisely induecd from short exact sequence $0 \to X \xrightarrow{\binom{u}{x}} Y \oplus I(X) \xrightarrow{(v,-\overline{u})} C_u \to 0$ in S. □

Let (B, S) and $(\mathsf{B}', \mathsf{S}')$ be Frobenius categories. An additive functor $F\colon \mathsf{B} \to \mathsf{B}'$ is called exact if $0 \to FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \to 0$ is contained in S' whenever $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is contained in S. If F transforms S-injectives into S'-injectives then F induces a functor $\underline{F}\colon \underline{\mathsf{B}} \to \underline{\mathsf{B}'}$. Denote by T (resp. T') the translation functor on $\underline{\mathsf{B}}$ (resp. $\underline{\mathsf{B}'}$).

Lemma 2.8. Let F be an exact functor between Frobenius categories B and B' such that F transforms S-injectives into S'-injectives. If there exists an invertible natural transformation $\alpha\colon \underline{F}T \to T'\underline{F}$ then $\underline{F}$ is an exact functor of triangulated categories.

Proof. Let $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ be an short exact sequence in S. Thanks to Lemma 2.7, it suffices to show that $\underline{F}$ sends a triangle induced by such a short exact sequence to a triangle in $\underline{\mathsf{B}'}$. Because F is exact, the image $0 \to FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \to 0$ is a short exact sequence in S'. By the Lemma 2.7 applied to $(\mathsf{B}', \mathsf{S}')$, this sequence induces a triangle in $\underline{\mathsf{B}'}$, $FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{-w'} T'FX$, where $w'\colon FZ \to T'FX$ is the connecting morphism constructed via an injective resolution of FX in B'. On the other hand we have a triangle $FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{-\alpha_X Fw} T'FX$. From Corollary 1.1 they are isomorphic. Therefore $\underline{F}$ preserves triangles. □