Ramsey with purple edges

Jingming Cui

2025-5-13

Ramsey theory is one of the most active areas in combinatorics. The classical Ramsey theory only considers coloring of a graph using two colors: red and blue. Now we will consider the case where some edges are colored with both red and blue, that is, purple. This problem inspired by an exercise posed by Angell in a mathematical magazine for high school students.

Question 1. Consider K_8, the complete graph on the vertex set $\{1, \cdots, 8\}$, such that the edges 12, 34, 56 and 78 are joined by a double edge, one coloured red and one coloured blue. Show that every colouring of the remaining (simple) edges using colours red and blue produces a red copy of K_3 or a blue copy of K_4.

A **red/blue/purple** colouring of K_n is a partition of the edges of K_n into three graphs $R \cup B \cup P$. We call a graph G H**-free** if G does not contain a copy of H as a subgraph. Given integers s, t, we call a red/blue/purple colouring $R \cup B \cup P$ (s, t)**-free** if $P \cup R$ is K_s-free and $P \cup B$ is K_t-free.

Definition 1. Given integers $s, t \geqslant 2$ and $n < r(s, t)$, let $g = g(n; s, t)$ denote the largest integer such that there exists an (s, t)-free red/blue/purple colouring $R \cup B \cup P$ of K_n that satisfies $|P| = g$. (Note that it is well-defined.)

It turns out that $g(n; s, t)$ is closely related to the so-called **Ramsey–Turán numbers**. For integers n, s and t, the Ramsey–Turán number $\mathbf{RT}_s(n, t)$ is the maximum number of edges in an n-vertex K_s-free graph G with independence number $\alpha(G) < t$.

Definition 2 (Blow-up colouring). Let $n \geqslant k \geqslant 1$ be integers and let G be a graph on k vertices $v_1, \cdots, v_k$. The n**-blowing-up colouring** of G is a colouring $R \cup B \cup P$ of K_n defined as follows. Let H be an equitable blow-up of G created by replacing every vertex $v_i \in V(G)$ by a set of vertices V_i of size either $\beta^- = \lfloor \frac{n}{k} \rfloor$ or $\beta^+ = \lceil \frac{n}{k} \rceil$, such that $\sum_i |V_i| = n$, and replacing every edge $v_i v_j$ of G by a complete bipartite graph between the sets V_i and V_j For each $i \in [k]$, denote the vertices of V_i by $\{u_1^i, \cdots, u_{\beta^-}^i\}$ or $\{u_1^i, \cdots, u_{\beta^+}^i\}$. For every edge $v_i v_j$ of G, and for every $l \in [\beta^+]$, colour the edges $u_l^i u_l^j$ of H, if it exists, red. Colour all other edges of H purple. Finally, colour all edges in the complement of H blue. Let R be the set of all red edges, B the set of all blue edges, and P the set of all purple edges.

Example 1. An example of red and purple edges in a 10-blow-up colouring of C_4.

Example 2. For $k \geqslant 2$, let Γ_k denote the Cayley graph over $\mathbb{Z}/(3k-1)\mathbb{Z}$ generated by the sum-free set $\{k, k+1, \cdots, 2k-1\}$. That is, for every $k \geqslant 2$, the vertex set of Γ_k is $\mathbb{Z}/(3k-1)\mathbb{Z}$ and its edge set is

$$E(\Gamma_k) = \left\{ ij \in \binom{V(\Gamma_k)}{2} : i - j \in \{k, k+1, \cdots, 2k-1\} \right\}.$$

For $kn/(3k-1) \leqslant t < (k-1)n/(3k-4)$, there is a "canonical" blow-up $\Gamma(n;k,t)$ of Γ_k by replacing vertices $1, k, 2k$ in Γ_k with sets of $(k-1)n - (3k-4)t$ vertices each, replacing the remaining $3k-4$ vertices in Γ_k with sets of $3t - n$ vertices, and replacing every edge uv of Γ_k by a complete bipartite graph. It is not hard to see $\alpha(\Gamma(n;k,t)) = t$ and

$$e(\Gamma(n;k,t)) = \frac{1}{2}k(k-1)n^2 - k(3k-4)tn + \frac{1}{2}(3k-4)(3k-1)t^2.$$

Now I will introduce some results of $g(n;s,t)$.

From Turán's Theorem we have the following estimate:

Theorem 1. For any $s \geqslant 3$, and $c > 1/(s-1)$, we have

$$g(n;s,cn) = (1 - O(1/n))\mathbf{RT}_s(n, sn)$$

Now we consider the triangle case.

Theorem 2. For every constant $0 < c \leqslant 1/3$, we have

$$g(n;3,cn) \geqslant \frac{n^2}{2}(c - o(1)).$$

Theorem 3. For every $c \in (\frac{1}{3}, \frac{1}{2})$ and positive integer n, we have

$$g(n;3,t+1) \geqslant e(\Gamma(n;k,t)) - O(n).$$

where $t = \lceil cn \rceil$ and k is the unique integer such that $kn/(3k-1) \leqslant t < (k-1)n/(3k-4)$.

Theorem 4. For all $c \in (0,1] \cup \left(\frac{1}{3}, \frac{4}{11}\right)$ we have

$$g(n;3,cn) = (1 - o(1))\mathbf{RT}_3(n, cn).$$

Theorem 5. For every $\gamma > 4$ there exists $n_0 \geqslant 1$ and $\delta \in (0, 1/2)$ such that for all $n \geqslant n_0$ and $t = \gamma\sqrt{n \log n}$ we have

$$g(n;3,t) \geqslant \delta\frac{nt}{2}.$$

Furthermore, if $t \gg \sqrt{n \log n}$ then

$$g(n;3,t) \geqslant \left(\frac{1}{2} - o(1)\right)\frac{nt}{2}.$$

Theorem 6. For every $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for n large enough and $t = (\sqrt{2} + \varepsilon)\sqrt{n \log n}$ we have $g(n; 3, t) \geqslant \delta nt/2$.

Now I will provide cases of general clique sizes.

Theorem 7. For any integer $l \geqslant 1$, and any constant $0 < c < \frac{1}{3l}$, we have

$$g(n; 2l+1, cn) \geqslant \left(\frac{l-1}{l} + c + o(1)\right)\frac{n^2}{2}.$$

Theorem 8. For any integer $l \geqslant 2$, and any constant $c < 1/(3l-2)$, we have

$$g(n; 2l, cn) \geqslant \frac{1}{2}\left(\frac{3l-5}{3l-2} + c - c^2 - o(1)\right)n^2.$$

By using probability method, mathematician have provided an upper bound of $g(n; s, t)$:

Theorem 9. There exists an absolute constant $p > 0$ such that for any fixed $s \geqslant 3$, for n large enough and any $t \in [R^{-1}(s, n) \log n, n]$, we have

$$p \cdot \mathbf{RT}_s(n, t/\log n) \leqslant g(n; s, t).$$

At the end of this note, I may provide some open problems.

More generally, let us denote by $g_M(n; s, t)$ the largest m such that there exists a red/blue/purple colouring of K_n with a matching of exactly m purple edges that has neither a red/purple copy of K_s nor a blue/purple copy of K_t. If, for example, $g_M(n; s, t) = 0$, then a single purple edge is sufficient to guarantee a red/purple K_s or a blue/purple K_t.

McKay provides the list $\mathcal{L}(n; s, t)$ of all graphs on $n = r(s, t) - 1$ vertices, with no clique of size s, and no independent set of size t, that is, the list of red graphs in (s, t)-free red/blue colourings of K_n. The results presented show that for all $t \leqslant 9$, a perfect matching of purple edges is sufficient to recover the Ramsey property in K_n. We wonder whether this holds for general t:

Problem 1. Is it true that for all integers $t \geqslant 3$ and $n = R(3, t) - 1$, every red/blue/purple colouring $R \cup B \cup P$ of K_n, where P is a perfect matching, contains a red/purple triangle or a blue/purple copy of K_t?

We cannot expect the answer to Problem 7.2 to be yes when 3 is replaced by $s \geqslant 4$. However, we would like to understand what happens when P contains some other, fixed structure. Natural candidates are Hamilton cycles.

Problem 2. Is it true that for all integers $s, t \geqslant 3$ and $n = R(s, t) - 1$, every red/blue/purple colouring $R \cup B \cup P$ of K_n, where P is a Hamilton cycle, contains a red/purple copy of K_s or a blue/purple copy of K_t?