Contents

Category Theory

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Historically, people first studied numbers and then realized that their collection forms a set: e.g. the set of natural numbers, rational numbers etc. The next natural question is as follows: what do sets or groups form? For two numbers they can either be equal or not equal. On the other hand for sets, besides equality one also has a notion of isomorphism. By abstracting the commonalities of families formed by objects of the same kind, we get the concept of a category.

Rather than characterize the objects directly, the categorical approach emphasizes the transformations between objects of the same general type. A fundamental lemma in category theory implies that any mathematical object can be characterized by its universal property—loosely by a representation of the morphisms to or from other objects of a similar form. Certain classes of universal properties define blueprints which specify how a new object may be built out of a collection of existing ones. Category theory also contributes new proof techniques, such as diagram chasing or arguments by duality; Steenrod called these methods "abstract nonsense."

In this note, we aim to establish the concept of a category. By using categorical language to generally describe various mathematical systems and mappings in these systems, we have a rough framework for the study of mathematical systems. In particular, categorical language is suitable for discussing homological algebra. It allows various concepts and structures within homological algebra to be simultaneously applied to a wide range of mathematical structures.

1 Categories, functors, Natural transformations

In this section, we'll provide some basic definitions of a category along with several examples.

1.1 Categories

Definition 1.1. A **category** C consists of the following data:

(1) a class of **objects**, denoted as $\mathrm{Ob}(\mathsf{C})$;

(2) for every two objects X, Y, a class $\mathrm{Hom}_{\mathsf{C}}(X,Y)$ (or $\mathrm{Hom}(X,Y)$, or $\mathrm{hom}_{\mathsf{C}}(X,Y)$, or $\mathrm{hom}(X,Y)$, or $\mathsf{C}(X,Y)$) whose elements are called **morphisms** and denoted by arrows; for example $f \colon X \to Y$

(3) a **composition rule** defined for morphisms: if $f \colon X \to Y$ and $g \colon Y \to Z$, then there is a morphism $gf \colon X \to Z$.

These data must satisfy the following two conditions:

(1) Composition is associative. That is, if $h \colon X \to Y$, $g \colon Y \to Z$, $f \colon Z \to W$, then $f(gh) = (fg)h$.

(2) There exist identity morphisms. That is, for every object X, there exists a morphism $\mathrm{id}_X \colon X \to X$ with the property that $f\,\mathrm{id}_X = f = \mathrm{id}_Y\, f$ whenever f is a morphism from X to Y.

Exercise. Verify that identity is unique.

Note that we don't assume that $\mathrm{Ob}(\mathsf{C})$ or $\mathrm{Hom}_{\mathsf{C}}(X,Y)$ are sets. When $\mathrm{Ob}(\mathsf{C})$ and all $\mathrm{Hom}_{\mathsf{C}}(X,Y)$ are both sets, we say C a **small category**; When $\mathrm{Hom}_{\mathsf{C}}(X,Y)$ are sets $\forall X, Y \in \mathrm{Ob}(\mathsf{C})$, we say that C is **locally small**.

Example 1.1. Here are some examples of category.

- Set: The objects are sets; the morphisms are functions; and composition is composition of functions.

- Set_*: The objects are sets with basepoints, e.g. (S, s_0) where $s_0 \in S$; the morphisms are functions respect to basepoints, i.e., a function $f \colon S \to T$ with $f(s_0) = t_0$; composition is composition of functions.

- Top: The objects are topological spaces; the morphisms are continuous functions; and composition is composition of functions.

- Grp: The objects are groups; the morphisms are group homomorphisms; and composition is composition of homomorphisms.

- Abel: The objects are abelian groups; the morphisms are group homomorphisms; and composition is composition of homomorphisms.

- Ring: The objects are rings; the morphisms are ring homomorphisms; and composition is composition of homomorphisms.
- CRing: The objects are commutative rings; the morphisms are ring homomorphisms; and composition is composition of homomorphisms.
- Vect_k: The objects are vector spaces over a field k; the morphisms are linear maps, and composition is composition of maps.
- Let R be a ring. R-Mod: The objects are left R-midules; the morphisms are module homomorphisms; and composition is composition of homomorphisms. Dually we define right R-module category Mod-R.
- Let $(X, \leq)$ be a poset. If we set $\mathrm{Ob}(\mathsf{C}) = P$, and $\forall a, b \in P$ define

$$\mathrm{Hom}_{\mathsf{C}}(a, b) := \begin{cases} \{*\} & \text{if } a \leq b \\ \emptyset & \text{if } a \not\leq b \end{cases}.$$

  It's easy to verify that it satisfies axioms of category. It is a small category.

Example 1.2. Let C be a category, Define its **opposite category** C^{op}: $\mathrm{Ob}(\mathsf{C}^{\mathrm{op}}) := \mathrm{Ob}(\mathsf{C})$, $\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(X, Y) := \mathrm{Hom}_{\mathsf{C}}(Y, X)$.

Example 1.3. Let C, D be categories. Define their **product category** $\mathsf{C} \times \mathsf{D}$: $\mathrm{Ob}(\mathsf{C} \times \mathsf{D}) := \mathrm{Ob}(\mathsf{C}) \times \mathrm{Ob}(\mathsf{D})$, $\mathrm{Hom}_{\mathsf{C}\times\mathsf{D}}((X, Y), (X', Y')) := \mathrm{Hom}_{\mathsf{C}}(X, X') \times \mathrm{Hom}_{\mathsf{D}}(Y, Y')$.

Definition 1.2. Let C be a category. D is a **subcategory** of C, if $\mathrm{Ob}(\mathsf{D}) \subseteq \mathrm{Ob}(\mathsf{C})$ and $\mathrm{Hom}_{\mathsf{D}}(X, Y) \subseteq \mathrm{Hom}_{\mathsf{C}}(X, Y)$ for all $X, Y \in \mathrm{Ob}(\mathsf{D})$.

We say that a subcategory is **full** if for all $X, Y \in \mathrm{Ob}(\mathsf{D})$, we have $\mathrm{Hom}_{\mathsf{D}}(X, Y) = \mathrm{Hom}_{\mathsf{C}}(X, Y)$.

Example 1.4. Abel is a full subcategory of Grp; CRing is a full subcategory of Ring; all finite-dim vector space forms a full subcategory of Vect_k.

Category theory is an appropriate setting in which to discuss an age-old question: "When are two objects really the same?" The concept of "sameness" being a special kind of relationship means that objects are the same if there is a particular morphism—an isomorphism—between them. How we talk about sameness is at the heart of category theory.

Definition 1.3. Let X and Y be objects in any category, and suppose $f\colon X \to Y$.

(1) f is **left invertible** if and only if there exists a morphism $g\colon Y \to X$ so that $gf = \mathrm{id}_X$. The morphism g is called a **left inverse** of f.

(2) f is **right invertible** if and only if there exists a morphism $h\colon Y \to X$ so that $fh = \mathrm{id}_Y$. The morphism h is called a **right inverse** of f.

(3) f is **invertible** and is said to be an **isomorphism** if it is both left and right invertible. Two objects X and Y are **isomorphic**, denoted $X \cong Y$, if there exists an isomorphism $f\colon X \to Y$.

In the case when f has both a left inverse g and a right inverse h, then

$$g = g\,\mathrm{id}_Y = gfh = \mathrm{id}_X\, h = h$$

and the single morphism $g = h$ is called the **inverse** of f.

Exercise. If f has an inverse, it is unique.

Exercise. The notion of being isomorphic is a form of equivalence, meaning it is reflexive, symmetric, and transitive. Isomorphic objects, therefore, form equivalence classes.

Here is a useful terminology:

Definition 1.4. For each morphism $f\colon X \to Y$ and object Z in a category, there is a map of sets $f_*\colon \operatorname{Hom}_{\mathsf{C}}(Z,X) \to \operatorname{Hom}_{\mathsf{C}}(Z,Y)$ called the **pushforward** of f defined by postcomposition $f_*\colon g \mapsto fg$.

$$\operatorname{Hom}_{\mathsf{C}}(Z,X) \xrightarrow{f_*} \operatorname{Hom}_{\mathsf{C}}(Z,Y)$$

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
Z \arrow[u, "g"] \arrow[ur, dashed, "f_*(g):=fg"']
\end{tikzcd}$$

There is also a map of sets $f^*\colon \operatorname{Hom}_{\mathsf{C}}(Y,Z) \to \operatorname{Hom}_{\mathsf{C}}(X,Z)$ called the **pullback** defined by precomposition $f^*\colon g \mapsto gf$.

$$\operatorname{Hom}_{\mathsf{C}}(X,Z) \xleftarrow{f_*} \operatorname{Hom}_{\mathsf{C}}(Y,Z)$$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, dashed, "f^*(g):=gf"'] & Y \arrow[d, "g"] \\
& Z
\end{tikzcd}$$

The reason for the visual names pushforward and pullback is hopefully clear from the diagrams above. With this terminology in hand, here is the theorem whose summary we provided above.

Theorem 1.1. Let C be a locally small category. TFAE:

(1) $f\colon X \to Y$ is an isomorphism;

(2) For every object Z, the pushforward $f_*\colon \mathrm{Hom}_{\mathsf{C}}(Z,X) \to \mathrm{Hom}_{\mathsf{C}}(Z,Y)$ is an isomorphism of sets.

(3) For every object Z, the pullback $f^*\colon \mathrm{Hom}_{\mathsf{C}}(Y,Z) \to \mathrm{Hom}_{\mathsf{C}}(X,Z)$ is an isomorphism of sets.

Proof. We just prove that a morphism $f\colon X \to Y$ is an isomorphism if and only if for every object Z, the pushforward $f_*\colon \mathrm{Hom}_{\mathsf{C}}(Z,X) \to \mathrm{Hom}_{\mathsf{C}}(Z,Y)$ is an isomorphism of sets.

Suppose $f\colon X \to Y$ is an isomorphism. Let $g\colon Y \to X$ be the inverse of f. Then for any Z, the map $g_*\colon \mathrm{Hom}_{\mathsf{C}}(Z,Y) \to \mathrm{Hom}_{\mathsf{C}}(Z,X)$ is the inverse of f_*.

Conversely, suppose that for any Z, the map $f_*\colon \mathrm{Hom}_{\mathsf{C}}(Z,X) \to \mathrm{Hom}_{\mathsf{C}}(Z,Y)$ is an isomorphism of sets. Choosing $Z = Y$, we have $f_*\colon \mathrm{Hom}_{\mathsf{C}}(Y,X) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{C}}(Y,Y)$, so in particular f_* is surjective. Therefore there exists a morphism $g\colon Y \to X$ so that $f_* g = \mathrm{id}_Y$, which by definition implies $fg = \mathrm{id}_Y$. To see that $gf = \mathrm{id}_X$, consider the case $Z = X$. We know $f_*\colon \mathrm{Hom}_{\mathsf{C}}(X,X) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{C}}(X,Y)$, so in particular f_* is injective. And since $f_*(\mathrm{id}_X) = f$ and $f_*(gf) = fgf = f$, injectivity of f_* implies $\mathrm{id}_X = gf$ as needed. $\square$

1.2 Functors

Definition 1.5. A **covariant functor** F from a category C to a category D consists of the following data:

(1) An object $F(X)$ of the category D for each object X in the category C;

(2) A morphism $F(f)\colon F(X) \to F(Y)$ for every morphism $f\colon X \to Y$;

These data must be compatible with composition and identity morphisms in the following sense:

(3) $(F(g))(F(f)) = F(gf)$ for any morphisms $f\colon X \to Y$ and $g\colon Y \to Z$;

(4) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any object X.

Definition 1.6. Dually, a functor $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ whose domain is called a **contravariant functor** form C to D.

Exercise. Let F be a functor. Show that F maps isomorphism to isomorphism.

Example 1.5. Here are some examples of functors.

- For an object X in a locally small category C, there is a functor $\mathrm{Hom}_{\mathsf{C}}(X,-)$ from C to Set that assigns to each object Z the set $\mathrm{Hom}_{\mathsf{C}}(X,Z)$ and to each morphism $f\colon Y \to Z$ the pushforward f_* of f as in definition 1.4.

$$\begin{tikzcd} Y \arrow[d, "f"'] & \longmapsto & \mathrm{Hom}_{\mathsf{C}}(X,Y) \arrow[d, "f_*"] \\ Z & & \mathrm{Hom}_{\mathsf{C}}(X,Z) \end{tikzcd}$$

- For an object X in a locally small category C, there is a functor $\mathrm{Hom}_{\mathsf{C}}(-,X)$ from C^{op} to Set that assigns to each object Z the set $\mathrm{Hom}_{\mathsf{C}}(Z,X)$ and to each morphism $f\colon Y \to Z$ the pullback f^* of f as in definition 1.4.

$$\begin{tikzcd} Y \arrow[d, "f"'] & \longmapsto & \mathrm{Hom}_{\mathsf{C}}(Y,X) \\ Z & & \mathrm{Hom}_{\mathsf{C}}(Z,X) \arrow[u, "f^*"'] \end{tikzcd}$$

- Fix a set X. There is a functor $X \times -$ from Set to Set defined on objects by $Y \mapsto X \times Y$ and on morphisms by $f \mapsto \mathrm{id} \times f$.

- There is a forgetful functor, usually denoted U for "underlying," from Grp to Set that forgets the group operation. Concretely, it sends a group G to its underlying set $U(G)$ and a group homomorphism to its underlying function.

- There is a free functor F from Set to Grp that assigns the free group $F(S)$ to the set S.

- Recall that for a ring R, a right R-module M and a left R-mod N, the tensor product of M and N over R, denoted $M \otimes_R N$, is an abelian group together with a product $\otimes\colon M \times N \to M \otimes_R N$ s.t. $m, m' \in M$, $n, n' \in N$ and $r \in R$ we have

  (1) $(m + m') \otimes n = m \otimes n + m' \otimes n$;

  (2) $m \otimes (n + n') = m \otimes n + m \otimes n'$;

  (3) $(m \cdot r) \otimes n = m \otimes (r \cdot n)$.

  In addition, $\otimes$ satisfy the universal property that if $f\colon M \times N \to G$ also satisfies (1)—(3), then f factors through $M \otimes_R N$:

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[dr, "f"'] & M \otimes_R N \arrow[d, dashed, "\exists! \tilde{f}"] \\ & G \end{tikzcd}$$

  Let R be the ring and B a left R-mod. Then $- \otimes_R B\colon \mathsf{Mod}\text{-}R \to \mathsf{Ab}$ becomes a functor.

Definition 1.7. Let F be a functor from a category C to a category D. If for all objects X and Y in C, the map $\mathrm{Hom}_{\mathsf{C}}(X,Y) \to \mathrm{Hom}_{\mathsf{D}}(F(X),F(Y))$ given by $f \mapsto F(f)$

(1) is injective, then F is called faithful;

(2) is surjective, then F is called full;

(3) is bijective, then F is called fully faithful.

1.3 Natural transformations

Definition 1.8. Let F and G be functors $\mathsf{C} \to \mathsf{D}$. A **natural transformation** η from F to G consists of a morphism $\eta_X \colon F(X) \to G(X)$ for each object X in C. Moreover, these morphisms in D must satisfy the property that $\eta_Y F(f) = G(f)\eta_X$ for every morphism $f \colon X \to Y$ in C. In other words, the following diagram must commute:

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

For any two functors $F, G \colon \mathsf{C} \to \mathsf{D}$, let $\mathsf{Nat}(F,G)$ denote the natural transformations from F to G. If $\exists \epsilon \colon G \to F$ s.t. $\epsilon \circ \eta = \mathrm{id}_F$ and $\eta \circ \epsilon = \mathrm{id}_G$, then η is called a **natural isomorphism** or a **natural equivalence**, and we say F and G are naturally isomorphic, denoted $F \cong G$.

Remark 1.1. A natural transformation is also called a morphism of functors.

Lemma 1.1. A natural transformation $\alpha \colon F \to G$ is an isomorphism of functors $\mathsf{C} \to \mathsf{D}$ iff $\alpha_X \colon F(X) \to G(X)$ is an isomorphism for all $X \in \mathrm{Ob}(\mathsf{C})$.

Proof. ($\Longrightarrow$) is clear.

($\Longleftarrow$) If each α_X is an isomorphism, define $\beta \colon G \to F$ by $\beta_X = \alpha_X^{-1}$. We need to check that β is a natural transformation. Given $f \colon X \to Y$, we know that

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\alpha_Y} & G(Y) \end{array}$$

$\Longrightarrow$

$$\begin{array}{ccc} G(X) & \xrightarrow{\beta_X} & F(X) \\ \downarrow{\scriptstyle G(f)} & & \downarrow{\scriptstyle F(f)} \\ G(Y) & \xrightarrow{\beta_Y} & F(Y) \end{array}$$

Hence β is a natural transformation. $\square$

Definition 1.9. Let C be a small category and D be any category. Define **functor category**

$$\mathsf{D}^{\mathsf{C}} := \mathsf{Fun}(\mathsf{C}, \mathsf{D})$$

to be the category whose objects are functors $F\colon \mathsf{C} \to \mathsf{D}$. Morphisms are natural transformations.

Definition 1.10. An **equivalence of categories** consists of functors $F\colon \mathsf{C} \rightleftarrows \mathsf{D}\colon G$, together with natural isomorphisms $\eta\colon 1_{\mathsf{C}} \cong GF$ and $\epsilon\colon FG \cong 1_{\mathsf{D}}$. Categories C and D are **equivalent**, written $\mathsf{C} \simeq \mathsf{D}$, if there exists an equivalence between them.

Exercise. The notion of equivalence of categories defines an equivalence relation.

Definition 1.11. A functor $F\colon \mathsf{C} \to \mathsf{D}$ is **essentially surjective** if for every object $Y \in \mathrm{Ob}(\mathsf{D})$ there is some $X \in \mathrm{Ob}(\mathsf{C})$ such that Y is isomorphic to $F(X)$.

Proposition 1.1. A functor $F\colon \mathsf{C} \to \mathsf{D}$ is a categorical equivalence iff it is fully faithful and essentially surjective.

Proof.

Lemma 1.2. Any morphism $f\colon X \to Y$ and fixed isomorphisms $X \cong X'$ and $Y \cong Y'$ determine a unique morphism $f'\colon X' \to Y'$ so that any of—or, equivalently, all of—the following four diagrams commute:

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "\cong"'] \arrow[d, "f'"] \\ Y \arrow[r, "\cong"'] & Y' \end{tikzcd} \quad \begin{tikzcd} X \arrow[r, "\cong"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "\cong"'] & Y' \end{tikzcd} \quad \begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "\cong"'] \arrow[d, "f'"] \\ Y & Y' \arrow[l, "\cong"] \end{tikzcd} \quad \begin{tikzcd} X \arrow[r, "\cong"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y & Y' \arrow[l, "\cong"] \end{tikzcd}$$

Proof of Lemma. The left-hand diagram defines f'. The commutativity of the remaining diagrams is left as exercise. □

Lemma 1.3. Consider morphisms with the indicated sources and targets

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[r, "j"] \arrow[d, "h"] & C \arrow[d, "l"] \\ A' \arrow[r, "k"'] & B' \arrow[r, "m"'] & C' \end{tikzcd}$$

and suppose that the outer rectangle commutes. This data defines a commutative rectangle if either:

(1) the right-hand square commutes and m is a monomorphism; or

(2) the left-hand square commutes and f is an epimorphism.

Proof of Lemma. The statements are dual. Assuming (1), $mkg = ljf = mhf$ by commutativity of the outer rectangle and right-hand square. Since m is a monomorphism, it follows that $kg = hf$ and thus that the diagram commutes. $\square$

First suppose that $F\colon \mathsf{C} \to \mathsf{D}$, $G\colon \mathsf{D} \to \mathsf{C}$, $\eta\colon 1_{\mathsf{C}} \cong GF$ and $\epsilon\colon FG \cong 1_{\mathsf{D}}$ define an equivalence of categories. For any $Y \in \mathrm{Ob}(\mathsf{D})$, the component of the natural isomorphism $\epsilon_Y\colon FG(Y) \cong Y$ demonstrates that F is essentially surjective. Consider a parallel pair $f, g\colon X \rightrightarrows X'$ in C. If $F(f) = F(g)$, then both f and g define an arrow $X \to X'$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X", "\cong"'] \arrow[d, "f \text{ or } g"'] & GF(X) \arrow[d, "GF(f)=GF(g)"] \\ X' \arrow[r, "\cong", "\eta_{X'}"'] & GF(X') \end{tikzcd}$$

that expresses the naturality of η commute. Lemma 1.2 implies that there is a unique arrow $X \to X'$ with this property, whence $f = g$. Thus, F is faithful, and by symmetry, so is G. Given $k\colon F(X) \to F(X')$, by Lemma 1.2, $G(k)$ and the isomorphisms η_X and $\eta_{X'}$ define a unique $h\colon X \to X'$ for which both $G(k)$ and $GF(h)$ make the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X", "\cong"'] \arrow[d, dashed, "h"'] & GF(X) \arrow[d, "G(k) \text{ or } GF(h)"] \\ X' \arrow[r, "\cong", "\eta_{X'}"'] & GF(X') \end{tikzcd}$$

commute. By Lemma 1.2 again, $GF(h) = G(k)$, whence $F(h) = k$ by faithfulness of G. Thus, F is fully faithful, and essentially surjective.

For the converse, suppose now that $F\colon \mathsf{C} \to \mathsf{D}$ is fully faithful and essentially surjective. Using essential surjectivity and the axiom of choice, choose, for each $Y \in \mathrm{Ob}(\mathsf{D})$, an object $G(Y) \in \mathrm{Ob}(\mathsf{C})$ and an isomorphism $\epsilon_Y\colon FG(Y) \cong Y$. For each $l\colon Y \to Y'$, Lemma 1.2 defines a unique morphism making the square

$$\begin{tikzcd} FG(Y) \arrow[r, "\epsilon_Y", "\cong"'] \arrow[d, dashed] & Y \arrow[d, "l"] \\ FG(Y') \arrow[r, "\epsilon_{Y'}", "\cong"'] & Y' \end{tikzcd}$$

commute. Since F is fully faithful, there is a unique morphism $G(Y) \to G(Y')$ with this image under F, which we define to be $G(l)$. This definition is arranged so that the chosen isomorphisms assemble into the components of a natural transformation $\epsilon\colon FG \to 1_{\mathsf{D}}$. It remains to prove that the assignment of arrows $l \mapsto G(l)$ is functorial and to define the natural isomorphism $\eta\colon 1_{\mathsf{C}} \to GF$.

Functorial property of G is another consequence of Lemma 1.2 and faithfulness of F. The morphisms $FG(1_Y)$ and $F(1_{G(Y)})$ both make

$$\begin{tikzcd}
FG(Y) \arrow[r, "\epsilon_Y", "\cong"'] \arrow[d, "FG(1_Y) \text{ or } F(1_{G(Y)})"'] & Y \arrow[d, "1_Y"] \\
FG(Y) \arrow[r, "\cong", "\epsilon_Y"'] & Y
\end{tikzcd}$$

commute, whence $G(1_Y) = 1_{G(Y)}$. Similarly, given $l' \colon Y' \to Y''$, both $F(G(l')G(l))$ and $FG(l'l)$ make

$$\begin{tikzcd}
FG(Y) \arrow[r, "\epsilon_Y", "\cong"'] \arrow[d, "F(G(l')G(l)) \text{ or } FG(l'l)"'] & Y \arrow[d, "l'l"] \\
FG(Y'') \arrow[r, "\cong", "\epsilon_{Y''}"'] & Y''
\end{tikzcd}$$

commute, whence $G(l')G(l) = G(l'l)$.

Finally, by full and faithfulness of F, we may define the isomorphisms $\eta_X \colon X \to GF(X)$ by specifying isomorphisms $F(\eta_X) \colon F(X) \to FGF(X)$. Define $F(\eta_X)$ to be $\epsilon^{-1}_{F(X)}$ For any $f \colon X \to X'$, the outer rectangle

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[d, "F(f)"'] & FGF(X) \arrow[r, "\epsilon_{F(X)}"] \arrow[d, "FGF(f)"'] & F(X) \arrow[d, "F(f)"] \\
F(X') \arrow[r, "F(\eta_{X'})"'] & FGF(X') \arrow[r, "\epsilon_{F(X')}"'] & F(X')
\end{tikzcd}$$

commutes, both composites being $F(f)$. The right-hand square commutes by naturality of ϵ. Because $\epsilon_{F(X')}$ is an isomorphism, this implies that the left-hand square commutes; see Lemma 1.3. Faithfulness of F tells us that $\eta_{X'} f = GF(f)\eta_X$, i.e., that η is a natural transformation. $\square$

Exercise. Let $\{*\}$ be a singleton set. Show that $\mathrm{Hom}_{\mathsf{Set}}(\{*\}, -) \colon \mathsf{Set} \to \mathsf{Set}$ be an equivalence of categories.

Exercise. Show that the category consists of all topological spaces and all maps is equivalent to Set.

Now we talk about the composition of natural transformations.

Definition 1.12. Given two categories C, D and a pair of functors $F, G \colon \mathsf{C} \to \mathsf{D}$ we depict a natural transformation $\eta \colon F \to G$ in the following way:

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathsf{D}
\arrow[Rightarrow, from=U, to=D, "\eta"]
\end{tikzcd}$$

There are two main ways one can compose natural transformations: horizontal and vertical.

(1) Consider a diagram

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=60, "F"] \arrow[r, "G"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3.5ex, "\Downarrow\eta"] \arrow[r, phantom, shift right=3.5ex, "\Downarrow\epsilon"] & \mathsf{D}
\end{tikzcd}$$

We have a **vertical composition** $F \to H$ which has components given by the composites $F(X) \xrightarrow{\eta_X} G(X) \xrightarrow{\epsilon_X} H(X)$ for all $X \in \mathrm{Ob}(\mathsf{C})$.

(2) Consider a diagram

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=50, "F_1"] \arrow[r, bend right=50, "F_2"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathsf{D} \arrow[r, bend left=50, "G_1"] \arrow[r, bend right=50, "G_2"'] \arrow[r, phantom, "\Downarrow\epsilon"] & \mathsf{E}
\end{tikzcd}$$

We have a **horizontal composition** $G_1F_1 \to G_2F_2$ which has components

$$G_1F_1(X) \xrightarrow{G_1(\eta_X)} G_1F_2(X) \xrightarrow{\epsilon_{F_2(X)}} G_2F_2(X)$$

or

$$G_1F_1(X) \xrightarrow{\epsilon_{F_1(X)}} G_2F_1(X) \xrightarrow{G_2(\eta_X)} G_2F_2(X)$$

for every $X \in \mathrm{Ob}(\mathsf{C})$. Note that we could have used the composition of ϵ and η in a different order; the two expressions coincide due to naturality of ϵ and η.

Theorem 1.2. (1) The vertical or horizontal composition of natural transformations is also a natural transformation.

(2) The vertical or horizontal composition satisfies associative law.

(3) For the diagram

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3.5ex, "\Downarrow\eta"] \arrow[r, phantom, shift right=3.5ex, "\Downarrow\epsilon"] & \mathsf{D} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3.5ex, "\Downarrow\eta'"] \arrow[r, phantom, shift right=3.5ex, "\Downarrow\epsilon'"] & \mathsf{E},
\end{tikzcd}$$

we have

$$\left(\epsilon' \underset{\text{ver}}{\circ} \eta'\right) \underset{\text{hor}}{\circ} \left(\epsilon \underset{\text{ver}}{\circ} \eta\right) = \left(\epsilon' \underset{\text{hor}}{\circ} \epsilon\right) \underset{\text{ver}}{\circ} \left(\eta' \underset{\text{hor}}{\circ} \eta\right)$$

Proof. (1) From the following diagram

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] \arrow[rr, bend left] & G(X) \arrow[r, "\epsilon_X"] \arrow[d, "G(f)"'] & H(X) \arrow[d, "H(f)"] \\
F(Y) \arrow[r, "\eta_Y"'] \arrow[rr, bend right] & G(Y) \arrow[r, "\epsilon_Y"'] & H(Y)
\end{tikzcd}$$

we have the naturality of vertical composition.

Now consider

$$\begin{tikzcd}
F_2F_1(X) \arrow[r, "F_2(\eta_X)"] \arrow[d, "F_2F_1(f)"'] \arrow[rr, bend left] & F_2G_1(X) \arrow[r, "\epsilon_{G_1(X)}"] \arrow[d, "F_2G_1(f)"'] & G_2G_1(X) \arrow[d, "G_2G_1(f)"] \\
F_2F_1(Y) \arrow[r, "F_2(\eta_Y)"'] \arrow[rr, bend right] & F_2G_1(Y) \arrow[r, "\epsilon_{G_1(Y)}"'] & G_2G_1(Y)
\end{tikzcd}$$

Since F_2 is a functor and η is a natural transformation, the left-hand square is commutative; since ϵ is a natural transformation, the right-hand square is commutative. Therefore the whole diagram is commutative, we have the naturality of horizontal composition.

(2) The associative law of vertical compositions is obvious form the associative law of morphisms.

Form the horizontal composition of natural transformations:

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathsf{D} \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\epsilon"] & \mathsf{E} \arrow[r, bend left, "H_1"] \arrow[r, bend right, "H_2"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathsf{F}
\end{tikzcd}$$

Then

$$\epsilon \underset{\text{hor}}{\circ} \eta \colon G_1F_1(X) \to G_2F_1(X) \to G_2F_2(X)$$

and

$$\theta \underset{\text{hor}}{\circ} \left(\epsilon \underset{\text{hor}}{\circ} \eta\right) : H_1(G_1F_1)(X) \to H_1(G_2F_2)(X) \to H_2(G_2F_2)(X)$$

implies

$$\theta \underset{\text{hor}}{\circ} \left(\epsilon \underset{\text{hor}}{\circ} \eta\right) : H_1G_1F_1(X) \to H_1G_2F_1(X) \to H_1G_2F_2(X) \to H_2G_2F_2(X);$$

Similarly

$$\left(\theta \underset{\text{hor}}{\circ} \epsilon\right) \underset{\text{hor}}{\circ} \eta \colon H_1G_1F_1(X) \to H_1G_2F_1(X) \to H_2G_2F_1(X) \to H_2G_2F_2(X)$$

$\forall X \in \mathrm{Ob}(\mathsf{C})$. Since θ is a natural transformation, we obtain

$$\begin{array}{ccc} H_1G_2F_1(X) & \longrightarrow & H_1G_2F_2(X) \\ \downarrow & & \downarrow \\ H_2G_2F_1(X) & \longrightarrow & H_2G_2F_2(X) \end{array}$$

and hence

$$\theta \underset{\text{hor}}{\circ} \left(\epsilon \underset{\text{hor}}{\circ} \eta\right) = \left(\theta \underset{\text{hor}}{\circ} \epsilon\right) \underset{\text{hor}}{\circ} \eta.$$

(3) For left-hand side,

$$\epsilon \underset{\text{ver}}{\circ} \eta \colon F_1(X) \to F_2(X) \to F_3(X),$$
$$\epsilon' \underset{\text{ver}}{\circ} \eta' \colon G_1(X) \to G_2(X) \to G_3(X),$$

and

$$\left(\epsilon' \underset{\text{ver}}{\circ} \eta'\right) \underset{\text{hor}}{\circ} \left(\epsilon \underset{\text{ver}}{\circ} \eta\right) : G_1F_1(X) \to G_1F_3(X) \to G_3F_3(X).$$

Hence

$$\left(\epsilon' \underset{\text{ver}}{\circ} \eta'\right) \underset{\text{hor}}{\circ} \left(\epsilon \underset{\text{ver}}{\circ} \eta\right) : G_1F_1(X) \to G_1F_2(X) \to G_1F_3(X) \to G_2F_3(X) \to G_3F_3(X).$$

For the right-hand side,

$$\eta' \underset{\text{hor}}{\circ} \eta \colon G_1F_1(X) \to G_1F_2(X) \to G_2F_2(X),$$
$$\epsilon' \underset{\text{hor}}{\circ} \epsilon \colon G_2F_2(X) \to G_2F_3(X) \to G_3F_3(X).$$

Hence

$$\left(\epsilon' \underset{\text{hor}}{\circ} \epsilon\right) \underset{\text{ver}}{\circ} \left(\eta' \underset{\text{hor}}{\circ} \eta\right) : G_1F_1(X) \to G_1F_2(X) \to G_1F_3(X) \to G_2F_3(X) \to G_3F_3(X).$$

Therefore

$$\left(\epsilon' \underset{\text{ver}}{\circ} \eta'\right) \underset{\text{hor}}{\circ} \left(\epsilon \underset{\text{ver}}{\circ} \eta\right) = \left(\epsilon' \underset{\text{hor}}{\circ} \epsilon\right) \underset{\text{ver}}{\circ} \left(\eta' \underset{\text{hor}}{\circ} \eta\right).$$

$\square$

In fact, we may consider morphisms between natural transformation, so we have the definition of n-category and even ∞-category.

Example 1.6. Let X be a topological space. A **presheaf** $\mathscr{F}$ of abelian groups on X consists of the data

(a) for every open subset $U \subseteq X$, an abelian group $\mathscr{F}(U)$, and

(b) for every inclusion $V \subseteq U$ of open subsets of X, a morphism of abelian groups $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$,

subject to the conditions

(1) $\mathscr{F}(\emptyset) = 0$, where $\emptyset$ is the emptyset set,

(2) ρ_{UU} is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$, and

(3) if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

For any topological space X, we define a category $\mathsf{Top}(X)$, whose objects are the open subsets of X, and where the only morphisms are the inclusion maps. Thus $\mathrm{Hom}(V, U)$ is empty if $V \not\subseteq U$, and $\mathrm{Hom}(V, U)$ has just one element if $V \subseteq U$. Now a presheaf is just a contravariant functor $\mathsf{Top}(X)^{\mathrm{op}} \to \mathsf{Abel}$.

We call the maps ρ_{UV} restriction maps, and we sometimes write $s|_V$ instead of $\rho_{UV}(s)$, if $s \in \mathscr{F}(U)$.

A presheaf $\mathscr{F}$ on a topological space X is a **sheaf** if it satisfies the following supplementary conditions:

(4) if U is an open set, if $\{V_i\}$ is an open covering of U, and if $s \in \mathscr{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i, then $s = 0$;

(5) if U is an open set, if $\{V_i\}$ is an open covering of U, and if we have elements $s_i \in \mathscr{F}(V_i)$ for each i, with the property that for each i, j, $s|_{V_i \cap V_j} = s|_{V_i \cap V_j}$ then there is an elements $s \in \mathscr{F}(U)$ such that $s|_{V_i}$ for each i. (Note condition (3) implies that s is unique.)

If $\mathscr{F}$ and $\mathscr{G}$ are presheaves on X, a **morphism** $\alpha \colon \mathscr{F} \to \mathscr{G}$ consists of a morphism of abelian groups $\alpha(U) \colon \mathscr{F} \to \mathscr{G}$ for each open set U, such that whenever $V \subseteq U$ is an inclusion, the diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\alpha(U)"] \arrow[d, "\rho_{UV}"] & \mathscr{G}(U) \arrow[d, "\rho'_{UV}"] \\ \mathscr{F}(V) \arrow[r, "\alpha(V)"] & \mathscr{G}(V) \end{tikzcd}$$

is commutative, where ρ and ρ' are the restriction maps in $\mathscr{F}$ and $\mathscr{G}$. This is a natural transformation form the functor $\mathscr{F}$ to $\mathscr{G}$. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves on X, we use the same definition for a morphism of sheaves. An **isomorphism** is a morphism which has a two-sided inverse.

Note that the composition of morphisms between presheaves $\mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3$ is precisely the vertical composition of natural transformations:

$$\begin{tikzcd}
\mathscr{F}_1(U) \arrow[r, "\alpha(U)"] \arrow[d] \arrow[rr, bend left] & \mathscr{F}_2(U) \arrow[r, "\beta(U)"] \arrow[d] & \mathscr{F}_3(U) \arrow[d] \\
\mathscr{F}_1(V) \arrow[r, "\alpha(V)"] \arrow[rr, bend right] & \mathscr{F}_2(V) \arrow[r, "\beta(V)"] & \mathscr{F}_3(V)
\end{tikzcd}$$

2 Yoneda Lemma

2.1 Yoneda Lemma

The language of natural transformations provides us an avenue in which comparing invariants (functors) becomes possible. The categorical philosophy that studying a mathematical object is more of a global, as opposed to a local, endeavor. That is, we can paint a better—rather, a complete—picture of an object once we investigate its interactions with all other objects. This theme finds its origins in an—if not the most—important result in category theory. Now we introduce Yoneda Lemma. It implies that one obejcts is determined by arrows between it and other objects.

Theorem 2.1 (Yoneda Lemma)**.** Let C be a locally small category. Then for every object X in C and for every contravariant functor $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$, the set of natural transformations from $\mathrm{Hom}_{\mathsf{C}}(-, X)$ to F is isomorphic to $F(X)$,

$$\mathsf{Nat}\left(\mathrm{Hom}_{\mathsf{C}}(-, X), F\right) \cong F(X).$$

Proof. Note that $\mathrm{Hom}_{\mathsf{C}}(X, X)$ has a special element namely id_X. So, for any natural transformation $\varphi\colon \mathrm{Hom}_{\mathsf{C}}(-, X) \to F$, one contains a special element $\varphi_X(\mathrm{id}_X) \in F(X)$, which completely determines φ. So it suffices to show how φ is uniquely determined.

We claim that for any $X \in \mathrm{Ob}(\mathsf{C})$ and $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$, the map

$$\begin{aligned} \psi\colon \mathsf{Nat}\left(\mathrm{Hom}_{\mathsf{C}}(-, X), F\right) &\longrightarrow F(X) \\ \varphi = \{\varphi_Y\colon \mathrm{Hom}_{\mathsf{C}}(Y, X) \to F(Y)\}_{Y \in \mathrm{Ob}(\mathsf{C})} &\longmapsto \varphi_X(\mathrm{id}_X) \end{aligned}$$

is a bijection of sets.

Now we construct the inverse map, given by

$$\begin{aligned}\phi\colon F(X) &\longrightarrow \mathsf{Nat}\left(\mathrm{Hom}_{\mathsf{C}}(-,X),F\right)\\ x &\longmapsto \phi(x)\end{aligned}$$

where $\phi(x)$ is given by

$$\begin{aligned}\phi(x)_Y\colon\ \mathrm{Hom}_{\mathsf{C}}(Y,X) &\longrightarrow F(Y)\\ (Y \xrightarrow{f} X) &\longmapsto Ff(x)\end{aligned}$$

There are three things to be checked.

(1) For every $x \in F(X)$, $\phi(x)$ is a natural transformation:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(Y,X) \arrow[r, "\phi(x)_Y"] \arrow[d, "g^*"] & F(Y) \arrow[d, "F(g)"] \\
\mathrm{Hom}_{\mathsf{C}}(Z,X) \arrow[r, "\phi(x)_Z"] & F(Z)
\end{tikzcd}$$

$$\begin{tikzcd}
(Y \xrightarrow{f} X) \arrow[r, mapsto] \arrow[d, mapsto] & Ff(x) \arrow[d, mapsto] & & & & \\
(Z \xrightarrow{g} Y \xrightarrow{f} X) \arrow[r, mapsto] & F(g^*f)(x) & = & F(f\circ g)(x) & = & Fg(Ff(x))
\end{tikzcd}$$

for any $g\colon Z \to Y$.

(2) $\psi\circ\phi = \mathrm{id}_{F(X)}$:

$$\psi\circ\phi(x) = \phi(x)_X(\mathrm{id}_X) = \mathrm{id}_{F(X)}(x) = x.$$

(3) $\phi\circ\psi = \mathrm{id}_{\mathsf{Nat}(\mathrm{Hom}_{\mathsf{C}}(-,X),F)}$:

$$\phi\circ\psi(\varphi) = \phi(\varphi_X(\mathrm{id}_X))$$

where

$$\begin{aligned}\phi(\varphi_X(\mathrm{id}_X))_Y\colon\ \mathrm{Hom}_{\mathsf{C}}(Y,X) &\longrightarrow F(Y)\\ (Y \xrightarrow{f} X) &\longmapsto Ff(\varphi_X(\mathrm{id}_X))\end{aligned}$$

By

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(X,X) \arrow[r, "\varphi_X"] \arrow[d, "f^*"] & F(X) \arrow[d, "F(f)"] \\
\mathrm{Hom}_{\mathsf{C}}(Y,X) \arrow[r, "\varphi_Y"] & F(Y)
\end{tikzcd}$$

$$\begin{tikzcd}
(X \xrightarrow{\mathrm{id}_X} X) \arrow[r, mapsto] \arrow[d, mapsto] & \varphi_X(\mathrm{id}_X) \arrow[d, mapsto] & \\
(Y \xrightarrow{f} X) \arrow[r, mapsto] & \varphi_Y(f) & = \quad Ff(\varphi_X(\mathrm{id}_X))
\end{tikzcd}$$

We see that $\phi \circ \psi(\varphi) = \varphi$.

$\square$

Corollary 2.1. Suppose C is a locally small category, then

$$\mathsf{Nat}\left(\mathrm{Hom}_{\mathsf{C}}(-, X), \mathrm{Hom}_{\mathsf{C}}(-, Y)\right) \cong \mathrm{Hom}_{\mathsf{C}}(X, Y).$$

Corollary 2.2. Suppose C is a locally small category, then $X \cong Y$ iff $\mathrm{Hom}_{\mathsf{C}}(-, X) \cong \mathrm{Hom}_{\mathsf{C}}(-, Y)$.

2.2 Representable functors

Definition 2.1. Let C be a locally small category, we say a functor $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is **representable** if there is a natural isomorphism $F \cong \mathrm{Hom}_{\mathsf{C}}(-, X)$ for some $X \in \mathrm{Ob}(\mathsf{C})$.

Remark 2.1. Form Yoneda Lemma, X in Definition 2.1 is unique up to an isomorphism.

Example 2.1. Here are some examples of representable functors.

- The identity functor $\mathrm{id}_{\mathsf{Set}}\colon \mathsf{Set} \to \mathsf{Set}$ is represented by the singleton set $\{*\}$. That is, for any set X, there is a natural isomorphism $\mathrm{Hom}_{\mathsf{Set}}(\{*\}, X) \cong X$ that defines a bijection between elements $x \in X$ and functions $x\colon \{*\} \to X$. Naturality says that for any $f\colon X \to Y$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Set}}(\{*\}, X) \arrow[r, "\cong"] \arrow[d, "f_*"'] & X \arrow[d, "f"] \\
\mathrm{Hom}_{\mathsf{Set}}(\{*\}, Y) \arrow[r, "\cong"'] & Y
\end{tikzcd}$$

  commutes, i.e., that the composite function $\{*\} \xrightarrow{x} X \xrightarrow{f} Y$ corresponds to the element $f(x) \in Y$, as is evidently the case.

- The forgetful functor $U\colon \mathsf{Grp} \to \mathsf{Set}$ is represented by the group $\mathbb{Z}$. That is, for any group G, there is a natural isomorphism $\mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G) \cong U(G)$ that associates, to every element $g \in U(G)$, the unique homomorphism $\mathbb{Z} \to G$ that maps the integer 1 to g. This defines a bijection because every homomorphism $\mathbb{Z} \to G$ is determined by the image of the generator 1; that is to say, $\mathbb{Z}$ is the free group on a single generator. This bijection is natural because the composite group homomorphism $\mathbb{Z} \xrightarrow{g} G \xrightarrow{\phi} H$ carries the integer 1 to $\phi(g) \in H$.

- The functor $U\colon \mathsf{Ring} \to \mathsf{Set}$ is represented by the unital ring $\mathbb{Z}[x]$, the polynomial ring in one variable with integer coefficients. A unital ring homomorphism $\mathbb{Z}[x] \to R$ is uniquely determined by the image of x.

- The forgetful functor $U\colon \mathsf{Top} \to \mathsf{Set}$ is represented by the singleton space: there is a natural bijection between elements of a topological space and continuous functions from the one-point space.

The Yoneda lemma also provides a more explicit characterization of the data encoded by a representation.

Definition 2.2. A **universal property** of an object $X \in \mathrm{Ob}(\mathsf{C})$ is expressed by a representable functor F together with a **universal element** $x \in F(X)$ that defines a natural isomorphism $\mathrm{Hom}_{\mathsf{C}}(X, -) \cong F$ or $\mathrm{Hom}_{\mathsf{C}}(-, X) \cong F$, as appropriate, via the Yoneda lemma.

Example 2.2. Recall from Example 2.1 that the forgetful functor $U\colon \mathsf{Ring} \to \mathsf{Set}$ is represented by the ring $\mathbb{Z}[x]$. The universal element, which defines the natural isomorphism

$$\mathrm{Hom}_{\mathsf{Ring}}(\mathbb{Z}[x], R) \cong UR$$

is the element $x \in \mathbb{Z}[x]$.

2.3 Adjoint functors

It was noted that category theory is an appropriate setting in which to discuss the concept of “sameness” between mathematical objects. This concept is captured by an isomorphism: a morphism from one object to another that is both left and right invertible. The discussion becomes especially interesting when those objects are categories. Two categories are isomorphic if there exists a pair of functors—one in each direction—whose compositions equal the identities. But equality is a lot of ask for! Isomorphisms of categories are too strict to be of much use. Relaxing the situation yields something better: categories C and D are equivalent if there exists a pair of functors $L\colon \mathsf{C} \rightleftarrows \mathsf{D}\colon R$ and natural isomorphisms $\mathrm{id}_{\mathsf{C}} \to RL$ and $LR \to \mathrm{id}_{\mathsf{D}}$.

Relaxing this a step further yields another gem of category theory: adjunctions. A pair of functors $L\colon \mathsf{C} \rightleftarrows \mathsf{D}\colon R$ forms an adjunction, and L and R are called adjoint functors, if there are natural transformations (not necessarily isomorphisms) $\eta\colon \mathrm{id}_{\mathsf{C}} \to RL$ and $\epsilon\colon LR \to \mathrm{id}_{\mathsf{D}}$ that, in addition, interact compatibly in a sense that can be made precise. Here the categories may not be equivalent, but don't think that adjunctions are mere second (or third) best. Quite often, relaxing a notion of equivalence results in a trove of rich mathematics. That is indeed the case here.

Definition 2.3. An **adjunction** consists of a pair of functors $L\colon \mathsf{C} \rightleftarrows \mathsf{D}\colon R$ and a natural bijection

$$\phi\colon \operatorname{Hom}_{\mathsf{D}}(L(X), Y) \cong \operatorname{Hom}_{\mathsf{C}}(X, R(Y)).$$

The functor L is called the **left adjoint** and the functor R is called the **right adjoint**, denoted as $L \dashv R$.

Remark 2.2. More precisely, the existence of natural isomorphism is equivalent to: $\forall X \in \operatorname{Ob}(\mathsf{C}), Y \in \operatorname{Ob}(\mathsf{D})$, there exists bijection $\phi_{X,Y}\colon \operatorname{Hom}_{\mathsf{D}}(L(X), Y) \to \operatorname{Hom}_{\mathsf{C}}(X, R(Y))$ s.t. for any morphism $f\colon X' \to X, g\colon Y' \to Y$, the following diagram commutes:

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathsf{D}}(L(X), Y') & \xrightarrow{\phi_{X,Y'}} & \operatorname{Hom}_{\mathsf{C}}(X, R(Y')) \\
{\scriptstyle g_*}\uparrow & & \uparrow{\scriptstyle (Rg)_*} \\
\operatorname{Hom}_{\mathsf{D}}(L(X), Y) & \xrightarrow{\phi_{X,Y}} & \operatorname{Hom}_{\mathsf{C}}(X, R(Y)) \\
{\scriptstyle (Lf)^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
\operatorname{Hom}_{\mathsf{D}}(L(X'), Y) & \xrightarrow{\phi_{X,Y'}} & \operatorname{Hom}_{\mathsf{C}}(X', R(Y))
\end{array}$$

The square at the top of the diagram shows that $\phi_{X,Y}$ is natural with respect to the variable Y, and the square at the bottom shows that $\phi_{X,Y}$ is natural with respect to the variable X.

Definition 2.4. Suppose $L\colon \mathsf{C} \rightleftarrows \mathsf{D}\colon R$ is an adjunction with adjunction isomorphism

$$\phi_{X,Y}\colon \operatorname{Hom}_{\mathsf{D}}(L(X), Y) \cong \operatorname{Hom}_{\mathsf{C}}(X, R(Y)).$$

By setting $Y = L(X)$, we have an isomorphism

$$\phi_{X,L(X)}\colon \operatorname{Hom}_{\mathsf{D}}\left(L(X), L(X)\right) \cong \operatorname{Hom}_{\mathsf{C}}\left(X, RL(X)\right)$$

Under this isomorphism, the morphism $\mathrm{id}_{L(X)}$ in the category D corresponds to a morphism $\eta_X \coloneqq \phi_{X,L(X)}\left(\mathrm{id}_{L(X)}\right) : X \to RL(X)$ in the category C. As one can check, these maps assemble into a natural transformation

$$\eta\colon \mathrm{id}_{\mathsf{C}} \to RL$$

called the **unit** of the adjunction. In other words, the unit is comprised of the adjuncts of the identity maps: $\eta_X \coloneqq \widehat{\mathrm{id}_{L(X)}}$.

Similarly for $X = R(Y)$, under the isomorphism

$$\phi_{R(Y),Y}\colon \operatorname{Hom}_{\mathsf{D}}\left(LR(Y), Y\right) \cong \operatorname{Hom}_{\mathsf{C}}\left(R(Y), R(Y)\right)$$

the morphism $\mathrm{id}_{R(Y)}$ in C corresponds to a morphism $\epsilon_Y := \widehat{\mathrm{id}_{R(Y)}}\colon LR(Y) \to Y$ that defines a natural transformation

$$\epsilon\colon LR \to \mathrm{id}_{\mathsf{D}}$$

called the **counit** of the adjunction.

Remark 2.3. More precisely, $\forall h\colon X' \to X$ in C, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{id}_{L(X)} \arrow[d, mapsto] & \in & \mathrm{Hom}_{\mathsf{D}}(L(X), L(X)) \arrow[r, "\phi"] \arrow[d, "L(h)^*"'] & \mathrm{Hom}_{\mathsf{C}}(X, RL(X)) \arrow[d, "h^*"] & \ni & \eta_X \arrow[d, mapsto] \\
L(h) & \in & \mathrm{Hom}_{\mathsf{D}}(L(X'), L(X)) \arrow[r, "\phi"] & \mathrm{Hom}_{\mathsf{C}}(X', RL(X)) & \ni & \phi(L(h)) \\
\mathrm{id}_{L(X')} \arrow[u, mapsto] & \in & \mathrm{Hom}_{\mathsf{D}}(L(X'), L(X')) \arrow[r, "\phi"'] \arrow[u, "(Lh)_*"] & \mathrm{Hom}_{\mathsf{C}}(X', RL(X')) \arrow[u, "(RL(h))_*"'] & \ni & \eta_{X'} \arrow[u, mapsto]
\end{tikzcd}$$

Then the following diagram commutes:

$$\begin{tikzcd}
X' \arrow[r, "\eta_{X'}"] \arrow[d, "h"'] \arrow[dr, "\phi(L(h))"] & RLX' \arrow[d, "RL(h)"] \\
X \arrow[r, "\eta_X"'] & RLX
\end{tikzcd}$$

This implies the naturality of η.

Meanwhile we have: $\forall f \in \mathrm{Hom}_{\mathsf{D}}(L(A), B)$, then $\phi_{A,B}(f) = R(f) \circ \eta_A$: From the naturality of ϕ we have

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{D}}(L(A), L(A)) \arrow[r, "\phi_{A,L(A)}"] \arrow[d, "f_*"'] & \mathrm{Hom}_{\mathsf{C}}(A, RL(A)) \arrow[d, "(R(f))_*"] \\
\mathrm{Hom}_{\mathsf{D}}(L(A), B) \arrow[r, "\phi_{A,B}"'] & \mathrm{Hom}_{\mathsf{C}}(A, R(B))
\end{tikzcd}$$

Since $\eta_A = \phi_{A,L(A)}(\mathrm{id}_{L(A)})$ form the definition of η, we obtain

$$\begin{tikzcd}
\mathrm{id}_{L(A)} \arrow[r, mapsto] \arrow[d, mapsto] & \eta_A \arrow[d, mapsto] \\
f \arrow[r, mapsto] & R(f) \circ \eta_A
\end{tikzcd}$$

Therefore $\phi_{A,B}(f) = R(f) \circ \eta_A$.

Similarly, $\forall g \in \mathrm{Hom}_{\mathsf{C}}(A, R(B))$, then $\phi_{A,B}^{-1}(g) = \epsilon_B \circ L(g)$.

Example 2.3. Here are several examples of adjoint functions.

- Let $\mathsf{Mod}\text{-}R$ be category of right R-modules. It's well-known that there exists a forgetful functor $U\colon \mathsf{Mod}\text{-}R \to \mathsf{Set}$; For a set X, it can generate a free right R-module $F(X) := \bigoplus_{x\in X} xR$, and $f\colon X \to Y$ could be naturally extended to $F(f)\colon F(X) \to F(Y)$. One may verify that F is a functor from Set to $\mathsf{Mod}\text{-}R$. Then we claim $F \dashv U$.

- There is a forgetful functor $U\colon \mathsf{Top} \to \mathsf{Set}$ that assigns to any topological space $(X, \mathcal{T}_X)$ the set X and to any continuous function $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ the function $f\colon X \to Y$. It is both a left and right adjoint in Top. Define a functor $D\colon \mathsf{Set} \to \mathsf{Top}$ that assigns to any set X the space $(X, \mathcal{T}_{\text{discrete}})$ with the discrete topology. To any function $f\colon X \to Y$, let $D(f) = f$, which is a continuous function. The setup $D\colon \mathsf{Set} \rightleftarrows \mathsf{Top}\colon U$ is an adjunction; for any set X and any space Y, we have

  $$\mathrm{Hom}_{\mathsf{Top}}(D(X), Y) \cong \mathrm{Hom}_{\mathsf{Set}}(X, U(Y))$$

  On the right, we have arbitrary functions from the set X into the space Y, viewed as a set. On the left, we take continuous functions $D(X) \to Y$, which are all functions from $X \to Y$, since every function from a discrete space is continuous.

  Here's another functor $I\colon \mathsf{Set} \to \mathsf{Top}$ that assigns to a set the same set with the indiscrete topology and to any function $f\colon X \to Y$, the same function, which will be continuous. Then $U\colon \mathsf{Top} \rightleftarrows \mathsf{Set}\colon I$ is an adjunction; for any space X and any set Y, we have

  $$\mathrm{Hom}_{\mathsf{Set}}(U(X), Y) \cong \mathrm{Hom}_{\mathsf{Top}}(X, I(Y))$$

  On the left we have arbitrary functions from X, viewed as a set, to the set Y. On the right, we have continuous functions $X \to I(Y)$, which are all functions $X \to Y$ since every function into an indiscrete space is continuous.

- If B is an R-S bimodule and C a right S-module, then $\mathrm{Hom}_{\mathsf{Mod}\text{-}R}(B, C)$ is naturally a right R-module by the rule $(fr)(b) = f(rb)$ for $f \in \mathrm{Hom}(B, C)$, $r \in R$ and $b \in B$.

  The functor $\mathrm{Hom}_{\mathsf{Mod}\text{-}S}(B, -)\colon \mathsf{Mod}\text{-}S \to \mathsf{Mod}\text{-}R$ is right adjoint to $- \otimes_R B$.

  That is for every R-module A and S-module C there is a natural isomorphism

  $$\tau\colon \mathrm{Hom}_S(A \otimes_R B, C) \xrightarrow{\sim} \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)).$$

  Proof. Given $f\colon A \otimes_R B \to C$, we define $(\tau f)(a)$ as the map $b \mapsto f(a \otimes b)$ for each $a \in A$. Given $g\colon A \to \mathrm{Hom}_S(B, C)$, we define $\sigma(g)$ to be the map $a \otimes b \mapsto g(a)(b)$. We have to check:

  (1) $(\tau f)(a)$ is an S-module map;

  (2) (τf) is an R-module map;

  (3) $\sigma(g)$ is an S-module map;

(4) $\sigma = \tau^{-1}$;

(5) τ is natural.

It's easy to check (1)—(3).

To see (4) is true, observe that $(\sigma\tau f)(a \otimes b) = (\tau f(a))(b) = f(a \otimes b)$ and $(\tau\sigma g(a))(b) = \sigma g(a \otimes b) = (g(a))(b) \Longrightarrow \sigma = \tau^{-1}$.

Next, we verify (5).

$$\begin{tikzcd}
\mathrm{Hom}_S(A' \otimes_R B, C) \arrow[r, "\tau"] \arrow[d, "(\varphi\otimes B)^*"'] & \mathrm{Hom}_R(A', \mathrm{Hom}_S(B, C)) \arrow[d, "\varphi^*"] \\
\mathrm{Hom}_S(A \otimes_R B, C) \arrow[r, "\tau"] \arrow[d, "\psi_*"'] & \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \arrow[d, "(\psi_*)_*"] \\
\mathrm{Hom}_S(A' \otimes_R B, C') \arrow[r, "\tau"'] & \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C'))
\end{tikzcd}$$

W.T.S.: Given an R-module map $\varphi\colon A \to A'$ and an S-module map $\psi\colon C \to C'$. If $f \in \mathrm{Hom}_S(A' \otimes_R B, C)$, then $\tau(\varphi \otimes B)^* f = \varphi^* \tau(f)$. Indeed,

$$\begin{aligned}
((\tau(\varphi \otimes B)^* f)(a))(b) &= ((\varphi \otimes B)^* f)(a \otimes b) \\
&= f(\varphi \otimes B)(a \otimes b) \\
&= f(\varphi(a) \otimes b)
\end{aligned}$$

and

$$\begin{aligned}
((\varphi^* \tau f)(a))(b) &= (\tau f(\varphi(a)))(b) \\
&= f(\varphi(a) \otimes b)
\end{aligned}$$

$\Longrightarrow$ the top square commutes.

For the bottom square, W.T.S.: if $f \in \mathrm{Hom}_S(A \otimes_R B, C)$, then $\tau\psi_*(f) = (\psi_*)_*\tau(f)$. Indeed,

$$(\tau\psi_* f(a))(b) = \psi_* f(a \otimes b) = \psi f(a \otimes b)$$

and

$$((\psi_*)_* \tau f(a))(b) = (\psi\tau f(a))(b) = \psi f(a \otimes b)$$

$\Longrightarrow$ The bottom square commutes. $\square$

Exercise. Let $L\colon \mathsf{C} \to \mathsf{D}$, $L'\colon \mathsf{D} \to \mathsf{E}$ is the left adjunction of $R\colon \mathsf{D} \to \mathsf{C}$, $R'\colon \mathsf{E} \to \mathsf{D}$. Show that $L' \circ L$ is the left adjunction of $R' \circ R$.

Definition 2.5. An adjunction between categories C and D is a pair of functors $L\colon \mathsf{C} \to \mathsf{D}$ and $R\colon \mathsf{D} \to \mathsf{C}$ together with natural transformations $\eta\colon \mathrm{id}_{\mathsf{C}} \to RL$ and $\epsilon\colon LR \to \mathrm{id}_{\mathsf{D}}$ such that for all objects $X \in \mathrm{Ob}(\mathsf{C})$ and $Y \in \mathrm{Ob}(\mathsf{D})$ the following triangles commute:

$$\begin{tikzcd} L(X) \arrow[r, "L(\eta_X)"] \arrow[dr, "\mathrm{id}_{L(X)}"'] & LRL(X) \arrow[d, "\epsilon_{L(X)}"] \\ & L(X) \end{tikzcd} \qquad \begin{tikzcd} R(Y) \arrow[r, "\eta_{R(Y)}"] \arrow[dr, "\mathrm{id}_{R(Y)}"'] & RLR(Y) \arrow[d, "R(\epsilon_Y)"] \\ & R(Y) \end{tikzcd}$$

Theorem 2.2. Definition 2.3 is equivalent to Definition 2.5.

Proof. First we show how η and ϵ in Definition 2.5 are determined by ϕ. We just need to prove the unit and counit satisfy the commutative diagram. From the definition of unit and Remark 2.3 we have $\mathrm{id}_{L(X)} = \phi^{-1}(\eta_X) = \epsilon_{L(X)} \circ L(\eta_X)$, and similarly $R(\epsilon_Y) \circ \eta_{R(Y)} = \mathrm{id}_{R(Y)}$.

Now we show how (η, ϵ) determine ϕ. Given $f\colon L(X) \to Y$, using $\phi(f) = R(f) \circ \eta_X$ directly to define ϕ, we only need to verify that this is an isomorphism. In fact, define $\psi(g) = \epsilon_Y \circ L(g)$, and there are three things to be checked:

(1) $\psi \circ \phi = \mathrm{id}$: Given $f\colon L(X) \to Y$,

$$\begin{aligned} \psi(\phi(f)) &= \psi\left(R(f) \circ \eta_X\right) \\ &= \epsilon_Y \circ L(R(f) \circ \eta_X) \\ &= \epsilon_Y \circ LR(f) \circ L(\eta_X) && \text{(The functorial property of } L\text{)} \\ &= f \circ \epsilon_{L(X)} \circ L(\eta_X) && \text{(The naturality of } \epsilon\text{)} \\ &= f \circ \mathrm{id}_{L(X)} && \text{(Definition 2.5)} \\ &= f. \end{aligned}$$

(2) $\phi \circ \psi = \mathrm{id}$: Similarly.

(3) ϕ is natural:

Fix X,

$$\begin{aligned} \phi_{X,Y'}(k \circ f) &= R(k \circ f) \circ \eta_X \\ &= R(k) \circ R(f) \circ \eta_X \\ &= R(k) \circ \phi_{X,Y}(f) \end{aligned}$$

implies the naturality of Y;

Fix Y,

$$\begin{aligned}
\phi_{X',Y}(f \circ L(h)) &= R(f \circ L(h)) \circ \eta_{X'} \\
&= R(f) \circ RL(h) \circ \eta_{X'} \\
&= R(f) \circ \eta_X \circ h \quad \text{(The naturality of } \eta) \\
&= \phi_{X,Y}(f) \circ h
\end{aligned}$$

implies the naturality of X.

□

Here are the universal property of adjunct functors:

Theorem 2.3. Given a pair of functors $L\colon \mathsf{C} \rightleftarrows \mathsf{D}\colon R$. Then $L \dashv R$ iff there exists natural transformation $\eta\colon \mathrm{id}_{\mathsf{C}} \to RL$ such that for any object $X \in \mathrm{Ob}(\mathsf{C})$ and morphism $f\colon A \to R(B)$, exists a unique $g\colon L(A) \to B$ s.t. the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & RL(A) \arrow[d, dashed, "R(g)"] \\
& R(B)
\end{tikzcd}$$

Proof. Assume that $L \dashv R$, we claim that the unit η satisfies the universal property. Form the natural isomorphism $\phi_{A,B}$ we can find a unique $g\colon L(A) \to B$ s.t. $\phi_{A,B}(g) = f$.

For the converse, assume that there is a natural transformation η satisfies the universal property. Let $A = R(Y)$ and $f = \mathrm{id}_{R(Y)}$, then the universal property gives a unique morphism $\epsilon_Y\colon LR(Y) \to Y$.

$$\begin{tikzcd}
R(Y) \arrow[r, "\eta_{R(Y)}"] \arrow[dr, "\mathrm{id}_{R(Y)}"'] & RLR(Y) \arrow[d, dashed, "R(\epsilon_Y)"] \\
& R(Y)
\end{tikzcd}$$

W.T.S. ϵ is natural and satisfies the condition in Definition 2.5.

Given a morphism $f\colon Y \to Y'$ in D. Apply the universal property on the morphism $R(h)\colon R(Y) \to R(Y')$, there is a unique $g\colon LR(Y) \to Y'$ s.t. $R(g) \circ \eta_{R(Y)} = R(h)$. Now

$$\begin{aligned}
R(\epsilon_{Y'} \circ LR(h)) \circ \eta_{R(Y)} &= R(\epsilon_{Y'}) \circ RLR(h) \circ \eta_{R(Y)} \\
&= R(\epsilon_{Y'}) \circ \eta_{R(Y')} \circ R(h) \quad \text{(The naturality of } \eta) \\
&= \mathrm{id}_{R(Y')} \circ R(h) \quad \text{(The definition of } \epsilon) \\
&= R(h),
\end{aligned}$$

and

$$\begin{aligned} R(h \circ \epsilon_Y) \circ \eta_{R(Y)} &= R(h) \circ R(\epsilon_Y) \circ \eta_{R(Y)} \\ &= R(h) \circ \mathrm{id}_{R(Y)} \\ &= R(h). \end{aligned}$$

From the uniqueness of g we have $\epsilon_{Y'} \circ LR(h) = k \circ \epsilon_Y$.

$\forall X \in \mathrm{Ob}(\mathsf{C})$, W.T.S. $\epsilon_{L(X)} \circ L(\eta_X) = \mathrm{id}_{L(X)}$. Let $B = L(X)$, there is a unique $g \colon L(X) \to L(X)$ s.t. $R(g) \circ \eta_X = \eta_X$.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\eta_X"'] & RL(X) \arrow[d, dashed, "R(g)"] \\ & RL(X) \end{tikzcd}$$

Obviously $g = \mathrm{id}_{L(X)}$ satisfies this equation. OOH we have

$$\begin{aligned} R(\epsilon_{L(X)} \circ L(\eta_X)) \circ \eta_X &= R(\epsilon_{L(X)}) \circ RL(\eta_X) \circ \eta_X \\ &= R(\epsilon_{L(X)}) \circ \eta_{RL(X)} \circ \eta_X && \text{(The naturality of } \eta) \\ &= \mathrm{id}_{RL(X)} \circ \eta_X && \text{(The definition of } \epsilon) \\ &= \eta_X. \end{aligned}$$

Therefore $\epsilon_{L(X)} \circ L(\eta_X) = \mathrm{id}_{L(X)}$ from the uniqueness. □

There is a close relationship between adjoint functors and representable functors.

Theorem 2.4. Let C, D be locally small categories. A functor $F \colon \mathsf{C} \to \mathsf{D}$ is a left adjunction iff $\forall Y \in \mathrm{Ob}(\mathsf{D})$, the functor $\mathrm{Hom}_{\mathsf{D}}(F(-), Y) \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is representable.

Proof. Let $G \colon \mathsf{D} \to \mathsf{C}$ is the right adjunction of F, $G(Y)$ is the representative of functor $\mathrm{Hom}_{\mathsf{D}}(F(-), Y)$.

$\forall Y \in \mathrm{Ob}(\mathsf{C})$, since $\mathrm{Hom}_{\mathsf{D}}(F(-), Y)$ is representable, there exists $G(Y) \in \mathrm{Ob}(\mathsf{C})$ and a natural isomorphism α_Y. For any morphism $f \colon Y \to Y'$ in D, consider the following commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{C}}(-, G(Y)) \arrow[r, "\cong"] \arrow[d, dashed] & \mathrm{Hom}_{\mathsf{D}}(F(-), Y) \arrow[d, "f \circ -"] \\ \mathrm{Hom}_{\mathsf{C}}(-, G(Y')) \arrow[r, "\cong"'] & \mathrm{Hom}_{\mathsf{D}}(F(-), Y') \end{tikzcd}$$

Lemma 1.2 gives the left-hand arrow. Hence we have a natural transformation $\mathrm{Hom}_{\mathsf{C}}(-, G(Y)) \to \mathrm{Hom}_{\mathsf{C}}(-, G(Y'))$. From the corollary of Yoneda Lemma we obtain a morphism $G(f) \colon G(Y) \to$

$G(Y')$. Therefore we have a functor G, and it's the right adjunction of F. The details are left as an exercise. $\square$

3 Limits and colimits

3.1 Diagrams are functors

Categorical limits and colimits are one of the—and in some sense the most—efficient way to build a new mathematical object from old objects. The constructions introduced in topology—subspaces, quotients, products, and coproducts—are examples, though the discussion can occur in any category. In fact, there are a number of other important constructions—pushouts, pullbacks, direct limits, and so on—so it's valuable to learn the general notion.

In topology, one asks for the limit of a sequence. In category theory, one asks for the (co)limit of a diagram. In what follows, it will be helpful to view a diagram as a functor. More specifically, a diagram in a category is a functor from the shape of the diagram to the category. For example, a commutative diagram like this

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

in a category C is a choice of three objects X, Y and Z and some morphisms $f\colon X \to Y$, $g\colon X \to Z$, and $h\colon Z \to Y$, with $f = hg$. It can be viewed as the image of a functor from an **indexing category**; that is, from a picture like this:

$$\begin{tikzcd} \bullet \arrow[rr] \arrow[dr] & & \bullet \\ & \bullet \arrow[ur] & \end{tikzcd}$$

We'll call the indexing categoey D. In summary,

$$\text{a diagram} \left(\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & Y \\ & Z \arrow[u, "h"'] \end{tikzcd} \in \mathsf{C} \right) \text{ is a functor } \left(\begin{tikzcd} \bullet \arrow[r] \arrow[dr] & \bullet \\ & \bullet \arrow[u] \end{tikzcd} \to \mathsf{C} \right)$$

Definition 3.1. Let D be a category. A **D-shaped diagram** in a category C is a functor $\mathsf{D} \to \mathsf{C}$. If the categories C and D are understood, we'll simply say **diagram** instead of D-shaped diagram in C.

Remark 3.1. When D is a small category, we call this a **small diagram**. This distinction is important for set-theoretic reasons when discussing limits and colimits. Most constructions in these notes are assumed to be small diagrams unless otherwise specified.

We can view any object A of C as a D-shaped diagram for any category D and namely as the constant functor. It is defined by sending every object in D to A and every arrow in D to the identity at A in C.

$$\begin{tikzcd} \bullet \arrow[d] & & A \arrow[d, "\mathrm{id}_A"] \\ \bullet & & A \arrow[from=1-1, to=2-3, phantom, "\xmapsto{A}"] \end{tikzcd}$$

Definition 3.2. Given a functor $F\colon \mathsf{D} \to \mathsf{C}$, a map from an object A to F—that is, an element of $\mathsf{Nat}\,(A, F)$—is called a **cone** from A to F. Similarly, a **cone** from F to A is an element of $\mathsf{Nat}\,(F, A)$.

Unwinding the definition, a cone from A to $F\colon \mathsf{D} \to \mathsf{C}$ is a collection of maps

$$\left\{A \xrightarrow{\eta_\bullet} F\bullet \ \text{ where } \bullet \text{ is an object in } \mathsf{D}\right\}$$

such that the diagrams:

$$\begin{tikzcd} & A \arrow[dl, "\eta_\bullet"'] \arrow[dr, "\eta_\circ"] & \\ F\bullet \arrow[rr, "F(\varphi)"] & & F\circ \end{tikzcd}$$

commute for every morphism $\bullet \xrightarrow{\varphi} \circ$ in D.

3.2 Limits and colimits

Here are the formal definitions of limit and colimit.

Definition 3.3. A **limit** of the diagram $F\colon \mathsf{D} \to \mathsf{C}$ is a cone η from an object $\lim F$ to the diagram satisfying the universal property that for any other cone γ from an object B to the diagram, there is a unique morphism $h\colon B \to \lim F$ such that $\gamma_\bullet = \eta_\bullet h$ for all objects $\bullet$ in D.

$$\begin{tikzcd} & B \arrow[dl, "\gamma_\bullet"'] \arrow[d, dashed, "h"] \\ F\bullet & \lim F \arrow[l, "\eta_\bullet"] \end{tikzcd}$$

Definition 3.4. A **colimit** of the diagram $F\colon \mathsf{D} \to \mathsf{C}$ is a cone from the diagram to an object $\operatorname{colim} F$ satisfying the universal property that for any other cone γ from the diagram to an object

B, there is a unique map $h\colon \operatorname{colim} F \to B$ such that $\gamma_\bullet = h\epsilon_\bullet$ for all objects $\bullet$ in D.

$$\begin{tikzcd} F\bullet \arrow[r, "\epsilon_\bullet"] \arrow[dr, "\gamma_\bullet"'] & \operatorname{colim} F \arrow[d, dashed, "h"] \\ & B \end{tikzcd}$$

Remark 3.2. We say a (co)limit is **small** if the diagram is small.

Remark 3.3. If the (co)limit of a diagram exists, it is unique up to a canonical isomorphism.

Remark 3.4. Given a diagram $F\colon \mathsf{E} \to \mathsf{C} \times \mathsf{D}$, it is equivalent to giving two diagrams $F_1\colon \mathsf{E} \to \mathsf{C}$ and $F_2\colon \mathsf{E} \to \mathsf{D}$. Moreover,

$$\lim F = (\lim F_1, \lim F_2)\,;$$
$$\operatorname{colim} F = (\operatorname{colim} F_1, \operatorname{colim} F_2)\,.$$

Depending on the shape of the indexing category, the (co)limit of a diagram may be given a familiar name: intersection, union, Cartesian product, kernel, direct sum, quotient, fibered product, and so on. The following examples illustrate this idea. In each case, recall that the data of a (co)limit is an object together with maps to or from that object, satisfying a universal property.

Definition 3.5. If the indexing category D is empty—no objects and no morphisms—then a functor $\mathsf{D} \to \mathsf{C}$ is an empty diagram. The limit of an empty diagram is called a **final object**. Dually, the colimit of an empty diagram is called an **initial object**.

Then the universal property of initial and final object is:

Proposition 3.1. We say that $X \in \mathrm{Ob}(\mathsf{C})$ is

(1) initial if $\forall Y \in \mathrm{Ob}(\mathsf{C})$, there exists a unique morphism $X \to Y$;

(2) final if $\forall Y \in \mathrm{Ob}(\mathsf{C})$, there exists a unique morphism $Y \to X$.

Definition 3.6. We say that $X \in \mathrm{Ob}(\mathsf{C})$ is **zero** if it is both initial and final, and denoted as 0.

Remark 3.5. If C has a 0, then $\forall X, Y \in \mathrm{Ob}(\mathsf{C})$,

$$\begin{tikzcd} & 0 \arrow[dr, dashed] & \\ X \arrow[ur, dashed] \arrow[rr, "0"'] & & Y \end{tikzcd}$$

$0\colon X \to Y$ is called the **zero morphism**.

Definition 3.7. If the indexing category D has no nonidentity morphisms—that is, if D is a discrete category—then a diagram $\mathsf{D} \to \mathsf{C}$ is just a collection of objects parametrized by D. In this case, the limit of the diagram is called the **product** and the colimit is called the **coproduct**.

Then the universal property of product and coproduct is:

Proposition 3.2. Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a family of objects in C.

The product of $\{X_\alpha\}$ consists of an object $Y \in \mathrm{Ob}(\mathsf{C})$ and $\pi_\alpha \colon Y \to X_\alpha$, $\alpha \in \Lambda$ such that $\forall W \in \mathrm{Ob}(\mathsf{C})$ and $f_\alpha \colon W \to X_\alpha$, $\alpha \in \Lambda$, $\exists! h \colon W \to Y$ s.t.

$$\begin{tikzcd} Y \arrow[r, "\pi_\alpha"] & X_\alpha \\ W \arrow[u, dashed, "h"] \arrow[ur, "f_\alpha"'] & \end{tikzcd} \quad , \alpha \in \Lambda$$

commutes. We denote Y by $\prod_{\alpha\in\Lambda} X_\alpha$.

Dually, the coproduct of $\{X_\alpha\}$ consists of an object $Z \in \mathrm{Ob}(\mathsf{C})$ and $\iota_\alpha \colon X_\alpha \to Z$, $\alpha \in \Lambda$ such that $\forall W \in \mathrm{Ob}(\mathsf{C})$ and $g_\alpha \colon X_\alpha \to W$, $\alpha \in \Lambda$, $\exists! k \colon Z \to W$ s.t.

$$\begin{tikzcd} X_\alpha \arrow[r, "\iota_\alpha"] \arrow[dr, "g_\alpha"'] & Z \arrow[d, dashed, "k"] \\ & W \end{tikzcd} \quad , \alpha \in \Lambda$$

commutes. We denote Z by $\coprod_{\alpha\in\Lambda} X_\alpha$.

Example 3.1. Here are some examples of initial/final objects and (co)products.

Category	initial/final object	product	coproduct
Set	$\emptyset$/singleton set	Descartes product	disjoint union
Top	$\emptyset$/point	product space	disjoint union
Grp	$\{1\}$	direct product	free product
Abel	0	direct product	direct sum
Ring	$\emptyset/0$	direct product	free product
CRing	$\mathbb{Z}/0$	direct product	tensor product
$(X, \leq)$	minimal/maximal element	infimum	supremum

Definition 3.8. A functor from $\bullet \to \bullet \leftarrow \bullet$ is a diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

and its limit is called the **pullback** of X and Y along the morphisms f and g. A square diagram decorated with a caret "$\lrcorner$" in the upper left corner denotes a pullback diagram. For example, this diagram

$$\begin{tikzcd} \circ \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

should be read as saying, "the square commutes and the object $\circ$ is the pullback."

Dually, a functor from $\bullet \leftarrow \bullet \rightarrow \bullet$ is a diagram,

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \\ Y & \end{tikzcd}$$

and its colimit is called the **pushout** of X and Y along the morphisms f and g. By analogy with pullbacks, we will use a caret to denote a pushout square. That is, a diagram such as

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[ur, phantom, very near end, "\ulcorner"] & \circ \end{tikzcd}$$

means that the square commutes and that $\circ$ is the pushout.

Pullback and pushout satisfy the following universal property:

Proposition 3.3. If W is the pullback of X and Y along the morphisms f and g, for any (S, α', β'), $\alpha' \colon S \to X$, $\beta' \colon S \to Y$ and $f\alpha' = g\beta'$, there is a unique morphism $\theta \colon S \to W$ such that $\alpha' = \alpha\theta$ and $\beta' = \beta\theta$.

$$\begin{tikzcd} S \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "\alpha'"] \arrow[ddr, bend right, "\beta'"'] & & \\ & W \arrow[r, "\alpha"] \arrow[d, "\beta"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"] & Z \end{tikzcd}$$

If W is the pushout of X and Y along the morphisms f and g, For any (S, α', β'), $\alpha' \colon X \to S$, $\beta' \colon Y \to S$ and $\alpha' f = \beta' g$, there is a unique morphism $\theta \colon W \to S$ such that $\alpha' = \theta\alpha$ and $\beta' = \theta\beta$.

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha"] \arrow[ddr, bend left, "\alpha'"] & \\ Y \arrow[r, "\beta"] \arrow[drr, bend right, "\beta'"'] & W \arrow[dr, dashed, "\theta"] & \\ & & S \end{tikzcd}$$

Definition 3.9. The limit of a diagram of the shape $\bullet \rightrightarrows \bullet$, such as

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

is called the **equalizer** of f and g.

Dually, the colimit of the same diagram

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

is called the **coequalizer** of f and g.

The universal property of equalizer and coequalizer are captured in the following two diagrams:

Proposition 3.4.

$$\begin{tikzcd} S \arrow[r, dashed] \arrow[rr, bend left] & E \arrow[r] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

where S is the equalizer and

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r] \arrow[rr, bend left] & C \arrow[r, dashed] & T \end{tikzcd}$$

where T is the coequalizer.

Definition 3.10. The limit of a diagram of the shape $\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots$, such as

$$X_\alpha \xleftarrow{f_\alpha} X_\beta \xleftarrow{f_\beta} X_\gamma \leftarrow \cdots$$

is called the **inverse limit** of the objects $\{X_\iota\}$ and denoted by $\varprojlim X_\iota$.

Dually, the colimit of a diagram of the shape $\bullet \to \bullet \to \bullet \to \cdots$, such as

$$X_\alpha \to X_\beta \to X_\gamma \to \cdots$$

is sometimes called the **directed limit** of the $\{X_\iota\}$. It is denoted by $\varinjlim X_\iota$.

Example 3.2. In categories $\mathsf{Set}, \mathsf{Top}, \mathsf{Grp}, \mathsf{Abel}, \mathsf{Ring}, \mathsf{CRing}$:

- Let D be a small category, $F\colon \mathsf{D} \to \mathsf{C}$, then

$$\lim F = \left\{ (x_\alpha) \in \prod_{\alpha \in \mathsf{Ob}(\mathsf{D})} F(\alpha) \mid F(t)(x_\alpha) = x_\beta,\ \forall t\colon \alpha \to \beta \text{ in } \mathsf{D} \right\}.$$

- The pullback of X and Y along the morphisms f and g is

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

- The equalizer of two morphisms f and g is

$$\{x \in X \mid f(x) = g(x)\}.$$

- Let D be a small category, $F\colon \mathsf{D} \to \mathsf{C}$, then the direct limit

$$\varinjlim F = \left(\coprod_{\alpha \in \mathrm{Ob}(\mathsf{D})} F(\alpha) \right) / \sim \text{ where } x_\alpha \sim x_\beta \iff \exists \gamma, t\colon \alpha \to \gamma, s\colon \beta \to \gamma \text{ s.t. } F(t)(x_\alpha) = F(s)(x_\beta).$$

- The coequalizer of two morphisms f and g is

$$Y/\sim \text{ where } \sim \text{ is the equivalence relation generated by } \{f(x) \sim g(x)\}_{x \in X}.$$

Exercise. Let A be a ring, then there is a equalizer diagram in Abel:

$$M \otimes A \otimes N \rightrightarrows M \otimes N \longrightarrow M \otimes_A N$$

Remark 3.6. (1) Let $F\colon \mathsf{D} \to \mathsf{C}$ be a diagram, $G\colon \mathsf{E} \to \mathsf{D}$ is a functor. Then there is a canonical morphism from $\lim F$ to $\lim FG$.

$$\begin{tikzcd} & \lim F \arrow[dl] \arrow[d, dashed] \\ FG\bullet & \lim FG \arrow[l] \end{tikzcd}$$

(2) Let $F, G\colon \mathsf{D} \to \mathsf{C}$ be two diagrams and α be a natural transformation $\alpha\colon F \to G$. Then α induces a morphism $\lim F \to \lim G$.

$$\begin{tikzcd} F\bullet \arrow[d, "\alpha_\bullet"'] & \lim F \arrow[l] \arrow[d, dashed] \\ G\bullet & \lim G \arrow[l] \end{tikzcd}$$

3.3 Properties of limits

Definition 3.11. A category is called **complete** if it contains the limits of small diagrams and is called **cocomplete** if it contains the colimits of all small diagrams.

Definition 3.12. A functor is **continuous** if and only if it takes limits to limits. It is **cocontinuous** if it takes colimits to colimits.

Theorem 3.1. Let C be a category, TFAE:

(1) C has all finite limits;

(2) C has finite limits and small inverse limits;

(3) C has small products and pullbacks;

(4) C has small products and equalizers.

Proof. (1) $\Longrightarrow$ (2): Trivial.

(2) $\Longrightarrow$ (3): Pullback is finite limit. So we just show how to construct small product. Let Λ be a small set, and all of its finite subset P under the inculsion relation $\supseteq$ forms a directed set. $\forall S \in P$, we have finite product $P_S = \prod_{\alpha \in S} X_\alpha$. Thus we have

$$\prod_{\alpha \in A} X_\alpha \cong \varprojlim_{S \in P} P_S.$$

(3) $\Longrightarrow$ (4): It's well-known that there is a diagonal morphism $\Delta \colon Y \to Y \times Y$. Let $(f, g) \colon X \to Y \times Y$ defined by $x \mapsto (f(x), g(x))$. Then the pullback E

$$\begin{tikzcd} E \arrow[r] \arrow[d] & Y \arrow[d, "\Delta"] \\ X \arrow[r, "(f,g)"'] & Y \times Y \end{tikzcd}$$

is the equalizer of f and g.

(4) $\Longrightarrow$ (1): Let D be a small category and functor $F \colon \mathsf{D} \to \mathsf{C}$ be a diagram. We define

$$P = \prod_{\alpha \in \mathrm{Ob}(\mathsf{D})} F(\alpha),$$
$$Q = \prod_{\substack{u \in \mathrm{Mor}(\mathsf{D}) \\ u \colon \alpha \to \beta}} F(\beta).$$

An arrow $P \to F(\beta)$ could be defined by

$$\phi_u \colon P \xrightarrow{\pi_\beta} F(\beta),$$
$$\psi_u \colon P \xrightarrow{\pi_\alpha} F(\alpha) \xrightarrow{F(u)} F(\beta).$$

Thus we obtain two arrows $P \to Q$. Then

$$\lim F = \text{the equalizer of } \phi \text{ and } \psi.$$

$\square$

Corollary 3.1. Assume the category C has all small limits and $F \colon \mathsf{C} \to \mathsf{D}$ be a functor. TFAE:

(1) F preserves all finite limits;

(2) F preserves finite limits and small inverse limits;

(3) F preserves small products and pullbacks;

(4) F preserves small products and equalizers.

Moreover we have

Theorem 3.2. Let C be a category. TFAE:

(1) C has all finite limits;

(2) C has final object and pullbacks;

(3) C has finite products and equalizers.

Assume the category C has all small limits and $F\colon \mathsf{C} \to \mathsf{D}$ be a functor. TFAE:

(1) F preserves all finite limits;

(2) F preserves final object and pullbacks;

(3) F preserves finite products and equalizers.

Now we have

Corollary 3.2. If a category has products and equalizers, then it is complete. If it has coproducts and coequalizers, then it is cocomplete.

Example 3.3. Categories $\mathsf{Set}, \mathsf{Top}, \mathsf{Grp}, \mathsf{Abel}, \mathsf{Ring}, \mathsf{CRing}$ have all small limits and colomits, because they have small products, equalizers, small coproducts and coequalizers.

Proposition 3.5. Let C be a locally small category and $X \in \mathrm{Ob}(\mathsf{C})$. Then the representable functors $\mathrm{Hom}_{\mathsf{C}}(-, X)\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ and $\mathrm{Hom}_{\mathsf{C}}(X, -)\colon \mathsf{C} \to \mathsf{Set}$ preserves all limits exists in C^{op} or C.

Proof. Let $F\colon \mathsf{D} \to \mathsf{C}$ be a diagram. W.T.S.

$$\mathrm{Hom}_{\mathsf{C}}(X, \lim F) \cong \lim \mathrm{Hom}_{\mathsf{C}}(X, F(\alpha)).$$

From the definition of $\lim F$ we have a collection of morphisms $\{\pi_\alpha\colon \lim F \to F(\alpha)\}_{\alpha \in \mathrm{Ob}(\mathsf{D})}$ s.t. $\forall f\colon \alpha \to \beta$ in D then $F(f) \circ \pi_\alpha = \pi_\beta$.

Define

$$\begin{aligned} \phi\colon \operatorname{Hom}_{\mathsf{C}}(X, \lim F) &\longrightarrow \lim \operatorname{Hom}_{\mathsf{C}}(X, F(\alpha)) \\ (g\colon X \to \lim F) &\longmapsto \{\pi_\alpha \circ g\}_{\alpha \in \mathrm{Ob}(\mathsf{D})} \end{aligned}$$

and from the universal property of limit we define

$$\begin{aligned} \psi\colon \lim \operatorname{Hom}_{\mathsf{C}}(X, F(\alpha)) &\longrightarrow \operatorname{Hom}_{\mathsf{C}}(X, \lim F) \\ \{h_\alpha\colon X \to F(\alpha)\} &\longmapsto \text{unique } h\colon X \to \lim F \end{aligned}$$

Obviously ϕ and ψ are both well-defined and they are inverse of each other.

Similarly, let $G\colon \mathsf{E} \to \mathsf{C}$ be a diagram, then

$$\operatorname{Hom}_{\mathsf{C}}(\operatorname{colim} G, X) \cong \lim \operatorname{Hom}_{\mathsf{C}}(G(\alpha), X).$$

$\square$

Example 3.4. The forgetful functor $\mathsf{Top}, \mathsf{Grp}, \mathsf{Abel}, \mathsf{Ring}, \mathsf{CRing} \to \mathsf{Set}$ preserves all small limits and small direct limits.

Note that all of these forgetful functors are representable and have left adjoints:

Theorem 3.3 (Adjoints and Limits Theorem)**.** Let $L\colon \mathsf{A} \to \mathsf{B}$ and $R\colon \mathsf{B} \to \mathsf{A}$ be an adjoint pair. Then

(1) L preserves all colimits. That is if $A\colon \mathsf{I} \to \mathsf{A}$ has a colimit, then so does $L(A)$ and $\operatorname{colim} L(A) = L(\operatorname{colim} A)$.

(2) R preserves all limits. That is if $B\colon \mathsf{J} \to \mathsf{B}$ has a limit, then so does $R(B)$ and $\lim R(B) = R(\lim B)$.

Proof. We only need to prove (1); applying the argument to R^{op} yield (2).

By the universal property of colimits,

$$\begin{tikzcd} L(A_i) \arrow[rr] \arrow[dr, "t_i"] \arrow[ddr, bend right, "L(s_i)"'] & & L(A_j) \arrow[dl, "t_j"'] \arrow[ddl, bend left, "L(s_j)"] \\ & \operatorname{colim} L(A) \arrow[d, shift right, "\beta_A"'] & \\ & L(\operatorname{colim} A) \arrow[u, dashed, shift right, "?\gamma"'] & \end{tikzcd}$$

commutes. W.T.S.: $\gamma\colon L(\operatorname{colim} A) \to \operatorname{colim} L(A)$ exists and γ is the inverse of β_A.

Consider applying R to the above diagram. Key: Note that we have the unit $\eta\colon 1_{\mathsf{A}} \to RL$ of the adjoint pair; this translates the diagram into a diagram in A.

$$\begin{tikzcd}
A_i \arrow[rr] \arrow[d, "\eta_{A_i}"] \arrow[ddr, bend right=60, "s_i"'] & & A_j \arrow[d, "\eta_{A_j}"'] \arrow[ddl, bend left=60, "s_j"] \\
RL(A_i) \arrow[rr] \arrow[dr] & & RL(A_j) \arrow[dl] \\
& R\operatorname{colim} L(A) & \\
& \operatorname{colim} A \arrow[u, dashed, "\tilde{\beta_A}"] &
\end{tikzcd}$$

The universal property of the $\operatorname{colim} A$ gives $\tilde{\beta_A}\colon \operatorname{colim} A \to R(\operatorname{colim} L(A))$. Since L and R are adjoint, we have a natural isomorphism

$$\tau\colon \operatorname{Hom}_{\mathsf{B}}(L(\operatorname{colim} A), \operatorname{colim} L(A)) \longrightarrow \operatorname{Hom}_{\mathsf{A}}(\operatorname{colim} A, R(\operatorname{colim} L(A))).$$

Let $\gamma = \tau^{-1}\tilde{\beta_A}$. It remains to show that

$$\begin{tikzcd}
L(A_i) \arrow[rr] \arrow[dr, "t_i"] \arrow[ddr, bend right=40, "L(s_i)"'] & & L(A_j) \arrow[dl, "t_j"'] \arrow[ddl, bend left=40, "L(s_j)"] \\
& \operatorname{colim} L(A) \arrow[d, shift right, "\beta_{\mathsf{A}}"'] & \\
& L(\operatorname{colim} A) \arrow[u, shift right, "\gamma"'] &
\end{tikzcd}$$

commutes. That is, $\gamma(Ls_i) = t_i$ for every $i \in \mathsf{I}$. Rewriting, we see

$$\gamma(Ls_i) = (\tau^{-1}\tilde{\beta_A})(Ls_i) = (Ls_i)^*(\tau^{-1}\tilde{\beta_A}) = \tau^{-1}(s_i)^*\tilde{\beta_A}.$$

Last "=" is because of the left square of the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathsf{B}}(L(A'), B) \arrow[r, "Lf^*"] \arrow[d, "\tau"] & \operatorname{Hom}_{\mathsf{B}}(L(A), B) \arrow[r, "g_*"] \arrow[d, "\tau"] & \operatorname{Hom}_{\mathsf{B}}(L(A'), B') \arrow[d, "\tau"] \\
\operatorname{Hom}_{\mathsf{A}}(A', R(B)) \arrow[r, "f^*"] & \operatorname{Hom}_{\mathsf{A}}(A, R(B)) \arrow[r, "Rg_*"] & \operatorname{Hom}_{\mathsf{A}}(A', R(B'))
\end{tikzcd}$$

We consider the term $\tau^{-1}(s_i)^*\tilde{\beta_A} = \tau^{-1}(\tilde{\beta_A}s_i)$. By the second diagram, $\tilde{\beta_A}s_i = Rt_i \circ \eta_{A_i}$. From Remark 2.3, we have $\tau(t_i) = R(t_i) \circ \eta_{A_i}$, and hence $\gamma(Ls_i) = \tau^{-1}(\tilde{\beta_A}s_i) = \tau^{-1}(Rt_i \circ \eta_{A_i}) = \tau^{-1}(\tau(t_i)) = t_i$. This is commutativity we desire.

Uniqueness of γ follows by the universal property of colimits. Hence, we have constructed a natural map $\gamma\colon L(\operatorname{colim} A) \to \operatorname{colim} L(A)$, which is the inverse of β_A, and we have a natural isomorphism $L(\operatorname{colim} A) \cong \operatorname{colim} L(A)$. $\square$

Theorem 3.4. Limits commute with limits in a category. That is, for a diagram $F\colon \mathsf{I}\times\mathsf{J}\to\mathsf{C}$, if the iterated limits

$$\lim_{\alpha\in\mathrm{Ob}(\mathsf{I})}\lim_{\beta\in\mathrm{Ob}(\mathsf{J})}F(\alpha,\beta),\qquad \lim_{\beta\in\mathrm{Ob}(\mathsf{J})}\lim_{\alpha\in\mathrm{Ob}(\mathsf{I})}F(\alpha,\beta)$$

both exist, then each is a limit of the diagram

$$\lim_{(\alpha,\beta)\in\mathrm{Ob}(\mathsf{I}\times\mathsf{J})}F(\alpha,\beta),$$

and hence they are canonically isomorphic.

Proof. From Proposition 3.5 we have

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{C}}\left(X,\lim_{\alpha\in\mathrm{Ob}(\mathsf{I})}\lim_{\beta\in\mathrm{Ob}(\mathsf{J})}F(\alpha,\beta)\right)&\cong\lim_{\alpha\in\mathrm{Ob}(\mathsf{I})}\lim_{\beta\in\mathrm{Ob}(\mathsf{J})}\mathrm{Hom}_{\mathsf{C}}\left(X,F(\alpha,\beta)\right);\\
\mathrm{Hom}_{\mathsf{C}}\left(X,\lim_{\beta\in\mathrm{Ob}(\mathsf{J})}\lim_{\alpha\in\mathrm{Ob}(\mathsf{I})}F(\alpha,\beta)\right)&\cong\lim_{\beta\in\mathrm{Ob}(\mathsf{J})}\lim_{\alpha\in\mathrm{Ob}(\mathsf{I})}\mathrm{Hom}_{\mathsf{C}}\left(X,F(\alpha,\beta)\right);\\
\mathrm{Hom}_{\mathsf{C}}\left(X,\lim_{(\alpha,\beta)\in\mathrm{Ob}(\mathsf{I}\times\mathsf{J})}F(\alpha,\beta)\right)&\cong\lim_{(\alpha,\beta)\in\mathrm{Ob}(\mathsf{I}\times\mathsf{J})}\mathrm{Hom}_{\mathsf{C}}\left(X,F(\alpha,\beta)\right).
\end{aligned}$$

In Set, one may directly verify that limits commute with limits, so

$$\lim_{\alpha\in\mathrm{Ob}(\mathsf{I})}\lim_{\beta\in\mathrm{Ob}(\mathsf{J})}\mathrm{Hom}_{\mathsf{C}}\left(X,F(\alpha,\beta)\right)\cong\lim_{\beta\in\mathrm{Ob}(\mathsf{J})}\lim_{\alpha\in\mathrm{Ob}(\mathsf{I})}\mathrm{Hom}_{\mathsf{C}}\left(X,F(\alpha,\beta)\right)\cong\lim_{(\alpha,\beta)\in\mathrm{Ob}(\mathsf{I}\times\mathsf{J})}\mathrm{Hom}_{\mathsf{C}}\left(X,F(\alpha,\beta)\right).$$

Therefore

$$\lim_{\alpha\in\mathrm{Ob}(\mathsf{I})}\lim_{\beta\in\mathrm{Ob}(\mathsf{J})}F(\alpha,\beta)\cong\lim_{\beta\in\mathrm{Ob}(\mathsf{J})}\lim_{\alpha\in\mathrm{Ob}(\mathsf{I})}F(\alpha,\beta)\cong\lim_{(\alpha,\beta)\in\mathrm{Ob}(\mathsf{I}\times\mathsf{J})}F(\alpha,\beta)$$

form the corollary of Yoneda Lemma. ◻

Theorem 3.5. In Set, small direct limit commutes with finite limit. That is, for a small direct set Λ, finite category I, and diagram $F\colon\Lambda\times\mathsf{I}\to\mathsf{Set}$, there exists a natural isomorphism

$$\varinjlim_{\alpha\in\Lambda}\lim_{\beta\in\mathrm{Ob}(\mathsf{I})}F(\alpha,\beta)\cong\lim_{\beta\in\mathrm{Ob}(\mathsf{I})}\varinjlim_{\alpha\in\Lambda}F(\alpha,\beta).$$

Proof. Since finite limits could be constructed by final object and pullbacks, we only show pullback commutes with small direct limit. Define

$$\begin{aligned}
\varinjlim_{\alpha\in\Lambda}\left(X_\alpha\times_{Z_\alpha}Y_\alpha\right)&\longrightarrow\left(\varinjlim_{\alpha\in\Lambda}X_\alpha\right)\times_{\left(\varinjlim_{\alpha\in\Lambda}Z_\alpha\right)}\left(\varinjlim_{\alpha\in\Lambda}Y_\alpha\right)\\
\left[(x_\alpha,y_\alpha)\right]&\longmapsto\left([x_\alpha],[y_\alpha]\right)
\end{aligned}$$

and it is a bijection. ◻

4 Abelian categories

Now we need to add new conditions to categories, so that we can establish homological algebra theory on these categories similar to abelian group categories or R-module categories. In the 1950s and early 1960s, much attention was focused on abelian group categories or R-module categories, became they provided a foundation for the development of homological algebra. On the other hand, B. Mitchell proved that every small abelian category can be embedded into a category of R-modules, and the exact relationships are preserved. Therefore, when we're discussing problems that only involve finite diagrams and concepts such as kernels, cokernels and images, we can consider that we're discussing them in the category R-modules. Hence some of our experience applies.

4.1 Additive categories and abelian categories

First we will generalize the concepts of monomorphisms and epimorphisms to general categories.

Definition 4.1. Let $f\colon X \to Y$ be a morphism in category C. If for all object $Z \in \mathrm{Ob}(\mathsf{C})$ and morphisms $g, h \in \mathrm{Hom}_{\mathsf{C}}(Z, X)$ with $fh = fg$, we have $h = g$, then we say f is a **monomorphism** or **monic**. If for all object $W \in \mathrm{Ob}(\mathsf{C})$ and morphisms $u, v \in \mathrm{Hom}_{\mathsf{C}}(Y, W)$ with $uf = vf$, we have $u = v$, then we say f is a **epimorphism** or **epic**.

Example 4.1. Here are some examples of monomorphisms and epimorphisms.

Category	isomorphism	monomorphism/epimorphism
Set	bijection	injection/surjection
Top	homeomorphism	continuous injection/surjection
Grp	group isomorphism	group monomorphism/epimorphism
Abel	group isomorphism	group monomorphism/epimorphism
Ring	ring isomorphism	ring monomorphism/-
CRing	ring isomorphism	ring monomorphism/-
$(X, \leq)$	$=$	all

Exercise. Show that the inclusive homomorphism $f\colon \mathbb{Z} \to \mathbb{Q}$ is epic in Ring, but it is not a ring epimorphism.

Exercise. Show that natural homomorphism $v\colon \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is monic in the category of divisible abelian groups, but it is not a group monomorphism.

Proposition 4.1. Let D be the pullback of B and C along the morphisms $f\colon B \to A$ and $g\colon C \to A$. Then g is monic iff $\beta\colon D \to B$ is monic; f is monic iff $\alpha\colon D \to C$ is monic.

Dually, if D is the pushout of B and C along the morphisms $f\colon A \to B$ and $g\colon A \to C$. Then f is epic iff $\beta\colon C \to D$ is epic; g is epic iff $\alpha\colon B \to D$ is epic.

Proof. We prove only the case of pullback.

Let D be the pullback of B and C along the morphisms $f\colon B \to A$ and $g\colon C \to A$.

Assume g is monic, and let $\gamma\colon E \to D$ s.t. $\beta\gamma = 0$. Then $0 = f\beta\gamma = g\alpha\gamma$, and we have $\alpha\gamma = 0$. From the definition of pullback we have $\gamma = 0$, and hence β is monic.

$$\begin{tikzcd}
E \arrow[dr, dashed, "\gamma"] \arrow[drr, "0"] \arrow[ddr, "0"'] & & \\
& D \arrow[r, "\alpha"] \arrow[d, "\beta"'] & C \arrow[d, "g"] \\
& B \arrow[r, "f"] & A
\end{tikzcd}$$

For the converse, let β be monic and $h\colon E \to C$ s.t. $gh = 0$. Moreover $f0 = 0$. Therefore exists a unique $\theta\colon E \to D$ with $h = \alpha\theta$ and $\beta\theta = 0$, so $\theta = 0$, so $h = 0$.

$$\begin{tikzcd}
E \arrow[dr, dashed, "\theta"] \arrow[drr, "h"] \arrow[ddr, "0"'] & & \\
& D \arrow[r, "\alpha"] \arrow[d, "\beta"'] & C \arrow[d, "g"] \\
& B \arrow[r, "f"] & A
\end{tikzcd}$$

□

Proposition 4.2. If

$$X \xrightarrow{f} Y \rightrightarrows Z$$

is a equalizer diagram, then f is monic; If

$$Y \rightrightarrows Z \xrightarrow{f} X$$

is a coequalizer diagram, then f is epic.

Proof. Exercise. □

Proposition 4.3. Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms in category C, then

(1) If f and g are monic, then gf is monic;

(2) If gf is monic, then f is monic;

(3) If f and g are epic, then gf is epic;

(4) If gf is epic, then g is epic;

(5) If f is an isomorphism, then it is both momic and epic.

Convention. We work with locally small categories.

Example 4.2. We know that $\mathsf{A} = R\text{-}\mathsf{Mod}$ has following properties:

(A1) Every hom-set $\mathrm{Hom}_{\mathsf{A}}(A, B)$ is an abelian group s.t. the composition

$$\mathrm{Hom}_{\mathsf{A}}(A, B) \times \mathrm{Hom}_{\mathsf{A}}(B, C) \to \mathrm{Hom}_{\mathsf{A}}(A, C)$$

is bilinear. E.g.

$$(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g;$$
$$f \circ (g_1 + g_2) = f \circ g_1 + f \circ g_2.$$

(A2) It has a zero object.

(A3) Any two objects have a product.

In addition,

(B1) Every map in A has a kernel and a cokernel.

(B2) Every injective map in A is the kernel of its cokernel

$$A \hookrightarrow B \twoheadrightarrow B/A .$$

every surjective map in A is the cokernel of its kernel

$$K \hookrightarrow A \twoheadrightarrow B .$$

(B3) Every map $f\colon A \to B$ factors as

$$A \xrightarrow{e} \mathrm{Im}(f) \xrightarrow{m} B$$

with e surjective and m injective.

Definition 4.2. An **additive category** A is a category satisfies:

(A1) Every hom-set $\mathrm{Hom}_{\mathsf{A}}(A, B)$ is an abelian group s.t. the composition

$$\mathrm{Hom}_{\mathsf{A}}(A, B) \times \mathrm{Hom}_{\mathsf{A}}(B, C) \to \mathrm{Hom}_{\mathsf{A}}(A, C)$$

is bilinear. E.g.

$$(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g;$$
$$f \circ (g_1 + g_2) = f \circ g_1 + f \circ g_2.$$

(A2) It has a zero object.

(A3) Any two objects have a product.

Definition 4.3. In any additive category A, a **kernel** of a morphism $f\colon A \to B$, denoted by $\ker(f)$ is defined to be a map $k\colon K \to A$ s.t. $fk = 0$ and that is universal w.r.t. this property

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!"'] \arrow[dr, "g"] \arrow[drr, bend left, "0"] & & \\ K \arrow[r, "k"] & A \arrow[r, "f"] & B \end{tikzcd}$$

In this case, we call K the **kernel object**, denoted by $\mathrm{Ker}(f)$.

Dually, a **cokernel** of f, denoted by $\mathrm{coker}(f)$ is a map $c\colon B \to C$, which is universal w.r.t. having $cf = 0$.

$$\begin{tikzcd} & & D \\ A \arrow[r, "f"] \arrow[urr, bend left, "0"] & B \arrow[r, "c"] \arrow[ur, "h"] & C \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

And C is called the **cokernel object**, denoted by $\mathrm{Coker}(f)$.

In fact, $\ker(f)$ is the equalizer of $(f, 0)$ and $\mathrm{coker}(f)$ is the coequalizer of $(f, 0)$. Hence:

Lemma 4.1. In any additive category A, every kernel is monic and every cokernel is epic.

Proof. Use Proposition 4.2. $\square$

Now, assume that A satisfies

(B1) Every morphism in A has a kernel and a cokernel.

Definition 4.4. In any additive category which satisfies (B1):

The **image** of a morphism $f\colon A \to B$ is defined as

$$\mathrm{im}(f) \coloneqq \ker(\mathrm{coker}(f))$$

and

$$\mathrm{Im}(f) \coloneqq \mathrm{Ker}(\mathrm{coker}(f)).$$

Similarly, the **coimage** of f is defined as

$$\mathrm{coim}(f) \coloneqq \mathrm{coker}(\ker(f))$$

and

$$\mathrm{Coim}(f) \coloneqq \mathrm{Coker}(\ker(f)).$$

Lemma 4.2. In any additive category which satisfies (B1), a morphism $f\colon A \to B$ can be factored uniquely as

$$A \longrightarrow \mathrm{Coim}(f) \longrightarrow \mathrm{Im}(f) \longrightarrow B\ .$$

Proof. There is a cononical map $u\colon \mathrm{Coim}(f) \to B$ because $f \circ \ker(f) = 0$.

$$\begin{tikzcd}
 & & \mathrm{Coim}(f) \arrow[d, dashed, "u"] & \\
K \arrow[r, "\ker(f)"] & A \arrow[ur, "\mathrm{coim}(f)"] \arrow[r, "f"] & B \arrow[r, "\mathrm{coker}(f)"] & C
\end{tikzcd}$$

The composition $\mathrm{coker}(f) \circ u = 0$, because $\mathrm{coker}(f) \circ u \circ \mathrm{coim}(f) = \mathrm{coker}(f) \circ f = 0$, and $\mathrm{coim}(f) = \mathrm{coker}(\ker(f))$ is epic. Hence, u factors uniquely through $\mathrm{im}(f)$, which gives us the desired map. $\square$

Lemma 4.3. In any additive category which satisfies (B1), for any morphism f, we have

$$\ker(\mathrm{coker}(\ker(f))) = \ker(f)$$

and

$$\mathrm{coker}(\ker(\mathrm{coker}(f))) = \mathrm{coker}(f).$$

Proof. We only prove that $\ker(\mathrm{coker}(\ker(f))) = \ker(f)$.

Let $k\colon K \to A$ be the kernel of f and $q\colon A \to C$ be the cokernel of k. Also, consider $k'\colon K' \to A$ as the kernel of the morphism q.

Given that $qk' = 0$, there exists a unique morphism $t\colon K \to K'$ s.t. $k = k't$.

$$\begin{tikzcd}
K \arrow[d, dashed, "t"'] \arrow[dr, "k"] \arrow[drr, bend left, "0"] & & \\
K' \arrow[r, "k'"] & A \arrow[r, "q"] & C
\end{tikzcd}$$

Since q is the cokernel of k and $fk = 0$, there exists a unique morphism $u\colon C \to B$ s.t. $f = uq$.

$$\begin{tikzcd} & & B \\ K \arrow[r, "k"] \arrow[urr, bend left, "0"] & A \arrow[r, "q"] \arrow[ur, "f"] & C \arrow[u, dashed] \end{tikzcd}$$

Then $fk' = uqk' = 0$. Since k is the kernel of $f\colon A \to B$, there exists a unique morphism $t'\colon K' \to K$ s.t. $k' = kt'$.

$$\begin{tikzcd} K' \arrow[d, dashed, "t'"'] \arrow[dr, "k'"] \arrow[drr, bend left, "0"] & & \\ K \arrow[r, "k"] & A \arrow[r, "f"] & B \end{tikzcd}$$

Observing that $k = k't$ and $k' = kt'$, it follows that $k = kt't$. Since k is monic, $t't = \mathrm{id}_{K'}$. Similarly we have $tt' = \mathrm{id}_K$. Therefore $K \cong K'$ and hence $\ker(q) = \ker(f)$. $\square$

Definition 4.5. An **abelian category** is an additive category A s.t.

(B1) Every morphism has a kernel and a cokernel.

(B2) Every monomorphism is the kernel of its cokernel; every epimorphism is the cokernel of its kernel.

Example 4.3. The category of finite abelian groups is abelian.

The category of free abelian groups is additive but NOT abelian, because it does not satisfy (B1).

Proof. Let $A = \langle a \rangle$ and $B = \langle b \rangle$ be free abelian groups. Consider the homomorphism defined by

$$\begin{aligned} \sigma\colon A &\longrightarrow B \\ na &\longmapsto 2nb \end{aligned}$$

Then σ is an monomorphism. Assume that the free abelian group category FAG forms an abelian category, then σ is the kernel of its cokernel. Let $\mathrm{Coker}(\sigma) = C$ and $\mathrm{coker}(\sigma) = \pi$,

$$A \xrightarrow{\sigma} B \xrightarrow{\pi} C$$

and we have $0 = \pi\sigma(a) = \pi(2b) = 2\pi(b) = 0 \Longrightarrow \pi = 0$. Then σ is an isomorphism, which leads to a contradiction. $\square$

Lemma 4.4. In an abelian category A, if f can be factored as $f = me$ s.t. m is monic and e is epic, then $e \cong \mathrm{coim}(f)$ and $m \cong \mathrm{im}(f)$.

Proof. We claim that $\ker(f) \cong \ker(e)$. Since $f\ker(e) = me\ker(e) = 0$, there exists a unique arrow $s\colon \mathrm{Ker}(e) \to \mathrm{Ker}(f)$ with $\ker(f)s = \ker(e)$.

$$\begin{tikzcd}
\mathrm{Ker}(e) \arrow[d, dashed, "s"'] \arrow[dr, "\ker(e)"] \arrow[drr, bend left, "0"] & & \\
\mathrm{Ker}(f) \arrow[r, "\ker(f)"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

OOH, because m is monic, $f\ker(f) = me\ker(f) = 0 \implies e\ker(f) = 0$. So there exists a unique arrow $t\colon \mathrm{Ker}(f) \to \mathrm{Ker}(e)$ with $\ker(e)t = \ker(f)$.

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[d, dashed, "t"'] \arrow[dr, "\ker(f)"] \arrow[drr, bend left, "0"] & & \\
\mathrm{Ker}(e) \arrow[r, "\ker(e)"] & A \arrow[r, "e"] & I
\end{tikzcd}$$

Obviously $st = \mathrm{id}$ and $ts = \mathrm{id}$, and so $\ker(f) = \ker(e)$. Hence, $e = \mathrm{coker}(\ker(e)) \cong \mathrm{coker}(\ker(f)) \cong \mathrm{coim}(f)$. Similarly, we have that $m \cong \mathrm{im}(f)$. □

Lemma 4.5. Let $f = me$, where $m = \mathrm{im}(f)$ and e epic. If also $f = m'e'$ s.t. m' is monic, then in the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "e'"'] & I \arrow[d, "m"] \arrow[dl, dashed, "t"'] \\
I' \arrow[r, "m'"'] & B
\end{tikzcd}$$

there is a unique arrow t with $m = m't$ and $e' = te$.

Proof. Since e is epic, $e = \mathrm{coker}(\ker(e))$. Now we claim that $e'\ker(e) = 0$, since $m'e'\ker(e) = me\ker(e) = 0$ and m' is monic. Form the universal of cokernel there exists a unique arrow $t\colon I \to I'$ s.t. $e' = te$.

$$\begin{tikzcd}
& & I' \\
\mathrm{Ker}(e) \arrow[r, "\ker(e)"] \arrow[urr, bend left, "0"] & A \arrow[r, "e"] \arrow[ur, "e'"] & I \arrow[u, dashed, "t"']
\end{tikzcd}$$

Since $f = me = m'e' = m'te$ and e is epic, we have $m = m't$. □

Lemma 4.5 is the universal property of image.

Proposition 4.4. In any abelian category, every morphism $f\colon A \to B$ can be uniquely factored as

$$A \xrightarrow{\mathrm{coim}(f)} C \xrightarrow{u} I \xrightarrow{\mathrm{im}(f)} B$$

where u is an isomorphism.

Proof. We only have to prove that u is an isomorphism. Let $m = \mathrm{im}(f)$. Then we may factor $f = me$. W.T.S. e is epic. Assume that $re = 0$, we want to prove that $r = 0$. e factors through $e = \ker(r)s$ and $f = m\ker(r)s$. Let $m' = m\ker(r)$, then m' is monic (Proposition 4.3 and Lemma 4.1). From Lemma 4.5, $\exists t$ with $m = m't = m\ker(r)t$ and hence $\ker(r)t = 1$. This means that the monomorphism $\ker(r)$ has a right inverse, so it is an isomorphism $\implies r = 0$. This proves that e is epic. So $e \cong \mathrm{coim}(f)$ (Lemma 4.4). Hence u is an isomorphism. $\square$

4.2 Exact functors

Definition 4.6. In an abelian category A, a sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of maps is called **exact at** B if $\ker(g) = \mathrm{im}(f)$. A sequence

$$\cdots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots$$

is called **exact** if it is exact at every X_i. An exact sequence of the form

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is called a **short exact sequence**.

Example 4.4. (1) A sequence $0 \longrightarrow X \xrightarrow{f} Y$ is exact iff f is monic;

(2) A sequence $0 \longrightarrow X \xrightarrow{f} Y \longrightarrow 0$ is exact iff f is an isomorphism;

(3) A sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact iff X is the kernel of g;

(4) For a morphism $f\colon X \to Y$, there is an exact sequence

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow X \xrightarrow{f} Y \longrightarrow \mathrm{Coker}(f) \longrightarrow 0 .$$

Definition 4.7. An **additive functor** $F\colon \mathsf{A} \to \mathsf{B}$ between additive categories A and B is a functor s.t. each $\mathrm{Hom}_{\mathsf{A}}(X,Y) \to \mathrm{Hom}_{\mathsf{B}}(F(X), F(Y))$ is a group homomorphism.

Definition 4.8. Let $F\colon \mathsf{A} \to \mathsf{B}$ be an additive functor between abelian categories.

If F is covariant, then F is called **left exact** (resp. **right exact**) if for every s.e.s.

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in A, the sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

(resp. $F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$) is exact in B.

If F is contravariant, then F is called **left exact** (resp. **right exact**) if for every s.e.s.

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in A, the sequence

$$0 \longrightarrow F(C) \longrightarrow F(B) \longrightarrow F(A)$$

(resp. $F(C) \longrightarrow F(B) \longrightarrow F(A) \longrightarrow 0$) is exact in B.

F is **exact** if it is both left and right exact.

Proposition 4.5. Let A be an abelian category. Then $\mathrm{Hom}_{\mathsf{A}}(M, -)$ is a left exact functor from A to Abel for every M in A. That is, given a s.e.s.

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in A, the sequence

$$0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C)$$

is exact.

Proof. If $\alpha \in \mathrm{Hom}(M, A)$, then $f_*(\alpha) = f\alpha$; If this is zero, then α must be zero since f is monic. Hence f_* is monic. Since $gf = 0$, we have $g_*(f_*(\alpha)) = gf\alpha = 0$, so $g_*f_* = 0$. It remains to show that if $\beta \in \mathrm{Hom}(M, B)$ s.t. $g_*(\beta) = g\beta$ is zero, then $\beta = f\alpha$ for some α. But if $g\beta = 0$, from the universal property of kernel β factors through A. $\square$

Corollary 4.1. $\mathrm{Hom}_{\mathsf{A}}(-, M)$ is a left exact contravariant functor.

Proof. Let

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a s.e.s. and we want to prove

$$0 \longrightarrow \mathrm{Hom}_R(C, M) \xrightarrow{\bar{g}} \mathrm{Hom}_R(B, M) \xrightarrow{\bar{f}} \mathrm{Hom}_R(A, M)$$

is exact.

Assume that $h \in \mathrm{Hom}(C, M)$ s.t. $\bar{g}(h) = hg = 0$. Since g is epic, we have $h = 0$, so $\bar{g}$ is monic. Let $h \in \mathrm{Hom}(C, M)$, we have $\bar{f}\bar{g}(h) = hgf = 0$, so $\mathrm{Im}(\bar{g}) \subseteq \mathrm{Ker}(\bar{f})$. If $h \in \mathrm{Ker}(\bar{f}) \subseteq \mathrm{Hom}(B, M)$, then $0 = \bar{f}(h) = hf$ and $0 = h(\mathrm{Im}(f)) = h(\mathrm{Ker}(g))$. Therefore there exists $\bar{h}\colon C \to M$ s.t. $h = \bar{h}g = \bar{g}(\bar{h})$. Hence $h \in \mathrm{Im}(\bar{g})$. $\square$

Theorem 4.1. Let $L\colon \mathsf{A} \to \mathsf{B}$ and $R\colon \mathsf{B} \to \mathsf{A}$ be an adjoint pair of additive functors. Then L is right exact and R is left exact.

Proof. Suppose that

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

is exact in B.

By naturality of τ , there is a commutative diagram for every A in A

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathsf{B}}(L(A), B') \arrow[r, "Lf^*"] \arrow[d, "\cong"', "\tau"] & \mathrm{Hom}_{\mathsf{B}}(L(A), B) \arrow[r, "g_*"] \arrow[d, "\cong"', "\tau"] & \mathrm{Hom}_{\mathsf{B}}(L(A), B'') \arrow[d, "\cong"', "\tau"] \\
0 \arrow[r] & \mathrm{Hom}_{\mathsf{A}}(A, R(B')) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathsf{A}}(A, R(B)) \arrow[r, "Rg_*"] & \mathrm{Hom}_{\mathsf{A}}(A, R(B''))
\end{tikzcd}$$

The top row is exact, because $\mathrm{Hom}(L(A), -)$ is left exact, so the bottom row is exact for all A. By the Yoneda Lemma, the Yoneda embedding reflects exactness, i.e. if for every M in A

$$\mathrm{Hom}_{\mathsf{A}}(M, A) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathsf{A}}(M, B) \xrightarrow{\beta_*} \mathrm{Hom}_{\mathsf{A}}(M, C)$$

is exact, then

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is exact. Hence, we know that

$$0 \longrightarrow R(B') \longrightarrow R(B) \longrightarrow R(B'')$$

must be exact. This proves that R is left exact. In particular, $L^{\mathrm{op}}\colon \mathsf{A}^{\mathrm{op}} \to \mathsf{B}^{\mathrm{op}}$ is left exact, i.e. L is right exact. $\square$

Theorem 4.2 (Freyd-Mitchell Embedding Theorem)**.** If A is a small category, then there is a ring R and an exact, fully faithful functor from A into R-Mod, which embeds A as a full subcategory of R-Mod.

Definition 4.9. A s.e.s. $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in abelian cateogry is called **split** if the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d, "\sim"] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] & X \oplus Z \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

An abelian category is called **semisimple** if all s.e.s. are split.

Example 4.5. Vect_k is semisimple, Abel is not abelian.

Definition 4.10. An object P in A is **projectiv**e if it satisfies the following universal lifting property: Given a epimorphism $g\colon B \to C$ and a $\gamma\colon P \to C$, there is at least one map $\beta\colon P \to B$ s.t. $\gamma = g\beta$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\beta"'] \arrow[d, "\gamma"] \\ B \arrow[r, two heads, "g"] & C \end{tikzcd}$$

Dually, an object I in A is **injective** if it satisfies the following universal lifting property: Given a monomorphism $f\colon A \to B$ and a map $\alpha\colon A \to I$, there exists at least one map $\beta\colon B \to I$ s.t. $\alpha = \beta f$.

$$\begin{tikzcd} A \arrow[r, hook, "f"] \arrow[d, "\alpha"] & B \arrow[dl, dashed, "\beta"] \\ I & \end{tikzcd}$$

Remark 4.1. An object M in A is projective (resp. injective) iff it is injective (resp. projective) as an object in A^{op}.

Lemma 4.6. P (resp. I) is a projective object (resp. an injective object) iff $\mathrm{Hom}_{\mathsf{A}}(P, -)$ (resp. $\mathrm{Hom}_{\mathsf{A}}(-, I)$) is exact.

Proof. For projective objects: suppose that $\mathrm{Hom}(M, -)$ is exact and that we can given a map $\gamma\colon M \to C$. We can lift $\gamma \in \mathrm{Hom}(M, C)$ to $\beta \in \mathrm{Hom}(M, B)$ s.t. $\gamma = g_*(\beta) = g\beta$ because g_* is onto. Thus M has the universal lifting property, i.e. it is projective.

Conversely, suppose M is projective. In order to show that $\mathrm{Hom}(M, -)$ is exact, it suffices to show that g_* is onto for every s.e.s.

$$0 \longrightarrow A \longrightarrow B \xrightarrow{g} C \longrightarrow 0$$

Given $\gamma \in \mathrm{Hom}(M, C)$, the universal lifting property of M gives $\beta \in \mathrm{Hom}(M, B)$, so that $\gamma = g\beta = g_*(\beta)$, i.e. g_* is onto. $\square$

Exercise. If $\{P_\alpha\}_{\alpha \in S}$ is a family of projective objects (resp. $\{I_\alpha\}_{\alpha \in S}$ is a family of injective objects), then $\bigoplus_{\alpha \in S} P_\alpha$ is projective (resp. $\prod_{\alpha \in S} I_\alpha$ is injective).

Proposition 4.6. If an additive functor $R\colon \mathsf{B} \to \mathsf{A}$ is right adjoint to an exact functor $L\colon \mathsf{A} \to \mathsf{B}$, then R preserves injectives, i.e. if I is an injective object of B, then $R(I)$ is injective in A.

Dually, if an additive functor $L\colon \mathsf{A} \to \mathsf{B}$ is left adjoint to an exact functor $R\colon \mathsf{B} \to \mathsf{A}$, then L preserves projectives.

Proof. W.T.S.: $\mathrm{Hom}_{\mathsf{A}}(-, R(I))$ is exact.

Given a monomorphism $f\colon A \to A'$ in A

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{B}}(L(A'), I) & \xrightarrow{Lf^*} & \mathrm{Hom}_{\mathsf{B}}(L(A), I) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_{\mathsf{B}}(A', R(I)) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathsf{B}}(A, R(I)) \end{array}$$

This diagram commutes by naturality of τ. Since L is exact and I is injective, the top map Lf^* is onto. Hence the bottom map f^* is onto, proving that $R(I)$ is an injective object. □

Definition 4.11. We say that A has **enough projectives** if for any object X of A, there is an epimorphism $P \to X$ with P projective.

We say that A has **enough injectives** if for every object X of A, there is a monomorphsim $X \to I$ with I injective.

Example 4.6. The category of finite abelian groups is an abelian category that has NO (nonzero) projective objects.

If $P \oplus Q$ is projective, then so is P. We also know that if P is projective, then any exact sequence of the form

$$A \longrightarrow P \longrightarrow 0$$

splits.

Now, by the Structure Theorem for abelian groups, it suffices to show that $\mathbb{Z}/p^n$ is not projective for any prime p and $n \geq 1$. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z}/p \xrightarrow{\iota} \mathbb{Z}/p^{n+1} \xrightarrow{\pi} \mathbb{Z}/p^n \longrightarrow 0 \ .$$

Since $\mathbb{Z}/p^{n+1}$ cannot be written as a direct sum of nontrivial cyclic groups, it follows that the above sequence does not split. Hence, $\mathbb{Z}/p^n$ is not projective.

We now introduce some stronger conditions for abelian categories.

Definition 4.12.(AB3) We say an abelian category satisfies (AB3) if it is cocomplete.

(AB3*) We say an abelian category satisfies (AB3*) if it is complete.

Definition 4.13.(AB4) We say a (AB3) category satisfies (AB4) if the direct sum of monomorphisms is also monic.

(AB4*) We say a (AB3*) category satisfies (AB4*) if it is the product of epimorphisms is also epic.

Definition 4.14. A nonempty category I is called **filtered** if

(1) For every $i, j \in \mathrm{Ob}(\mathsf{I})$, there are arrows

$$i \longrightarrow k\ , \quad j \longrightarrow k$$

to $k \in \mathrm{Ob}(\mathsf{I})$.

(2) For every two parallel arrows

$$u, v\colon\ i \rightrightarrows j$$

there is an arrow $w\colon j \to k$ s.t. $wu = wv\colon i \to k$.

A **filtered colimit** in A is just the colimit of a functor $A\colon \mathsf{I} \to \mathsf{A}$ in which I is a filtered category.

Definition 4.15.(AB5) We say a (AB3) category satisfies (AB5) if the filtered colimit of s.e.s. is also a s.e.s.

(AB5*) We say a (AB3) category satisfies (AB5*) if the filtered limit of s.e.s. is also a s.e.s.

4.3 A baby vision of derived category

Before introducing derived categories, we first recall the basic notions of chain complexes in abelian categories. In this subsection, we fix A be an abelian category.

Definition 4.16. A **chain complex** $C_\bullet$ in the category A is a family $\{C_n\}_{n\in\mathbb{Z}}$ of objects, together with morphisms $d = d_n\colon C_n \to C_{n-1}$ s.t. $dd\colon C_n \to C_{n-2}$ is zero. The maps d_n are called **differentials** of $C_\bullet$.

Lemma 4.7. Given a commutative diagram in A

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, "c"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \end{tikzcd}\ , \quad g'f' = 0, \quad gf = 0,$$

then there exists a unique morphism $\Phi\colon H[X \to Y \to Z] \to H[X' \to Y' \to Z']$ makes the following diagram commutes

$$\begin{tikzcd} \mathrm{Ker}(g) \arrow[r, two heads] \arrow[d] & H[X \to Y \to Z] \arrow[r, hook] \arrow[d, "\Phi"] & \mathrm{Coker}(f) \arrow[d] \\ \mathrm{Ker}(g') \arrow[r, two heads] & H[X' \to Y' \to Z'] \arrow[r, hook] & \mathrm{Coker}(f') \end{tikzcd}$$

The morphism $\mathrm{Ker}(g) \to \mathrm{Ker}(g')$ is induced by the universal property of kernel

$$\begin{tikzcd} \mathrm{Ker}(g) \arrow[r, hook, "\ker(g)"] \arrow[d, dashed] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[d, "c"] \\ \mathrm{Ker}(g') \arrow[r, hook, "\ker(g')"] & Y' \arrow[r, "g'"'] & Z' \end{tikzcd}$$

because $g'b\ker(g) = cg\ker(g) = 0$. $\mathrm{Coker}(f) \to \mathrm{Coker}(f')$ is constructed similarly.

Proof. Since $\mathrm{Im}(g) = \mathrm{Ker}(Z \to \mathrm{Coker}(g))$ and $\mathrm{Im}(g') = \mathrm{Ker}(Z' \to \mathrm{Coker}(g'))$, we have the morphism $\mathrm{Im}(g) \to \mathrm{Im}(g')$ from the universal property of kernel:

$$\begin{tikzcd}
\mathrm{Im}(g) \arrow[r, hook] \arrow[d, dashed] & Z \arrow[r, two heads] \arrow[d, "c"] & \mathrm{Coker}(g) \arrow[d] \\
\mathrm{Im}(g') \arrow[r, hook] & Z' \arrow[r, two heads] & \mathrm{Coker}(g')
\end{tikzcd}$$

The morphism $g\colon Y \to Z$ can be factored through $\mathrm{Im}(g)$,

$$\begin{tikzcd}
Y \arrow[r, "e"] & \mathrm{Im}(g) \arrow[r, hook] & Z
\end{tikzcd}.$$

We have $ef = 0$ because $\mathrm{im}(g)ef = 0$ and $\mathrm{im}(g)$ is monic. From the universal property of cokernel, we obtain a unique arrow $\psi\colon \mathrm{Coker}(f) \to \mathrm{Im}(g)$,

$$\begin{tikzcd}
 & & \mathrm{Im}(g) \\
X \arrow[r, "f"] \arrow[urr, bend left, "0"] & Y \arrow[r, two heads] \arrow[ur, "e"] & \mathrm{Coker}(f) \arrow[u, dashed, "\psi"']
\end{tikzcd}$$

Moreover, $Y \to \mathrm{Coker}(f) \to \mathrm{Im}(g)$ is epic, the morphism ψ is epic thanks to Proposition 4.3. Similarly we have $\mathrm{Coker}(f') \to \mathrm{Im}(g')$. Now consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, two heads, "\mathrm{coker}(f)"] \arrow[d, "b"] & \mathrm{Coker}(f) \arrow[r, two heads, "\psi"] \arrow[d, "\mathrm{coker}(b)"] & \mathrm{Im}(g) \arrow[r, hook, "\mathrm{im}(g)"] \arrow[d, "\mathrm{im}(c)"] & Z \arrow[d, "c"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, two heads, "\mathrm{coker}(f')"'] & \mathrm{Coker}(f') \arrow[r, two heads, "\psi'"'] & \mathrm{Im}(g') \arrow[r, hook, "\mathrm{im}(g')"'] & Z
\end{tikzcd}$$

Note that $cg = g'b$ implies $c\,\mathrm{im}(g)\psi\,\mathrm{coker}(f) = \mathrm{im}(g')\psi'\,\mathrm{coker}(f')b$ and $c\,\mathrm{im}(g) = \mathrm{im}(g')\,\mathrm{im}(c)$, we have $\mathrm{im}(g')\,\mathrm{im}(c)\psi\,\mathrm{coker}(f) = \mathrm{im}(g')\psi'\,\mathrm{coker}(f')b$, and so $\mathrm{im}(c)\psi\,\mathrm{coker}(f) = \psi'\,\mathrm{coker}(f')b$ (here $\mathrm{im}(g')$ is monic). Since $\mathrm{coker}(b)\,\mathrm{coker}(f) = \mathrm{coker}(f')b$, we have $\mathrm{im}(c)\psi\,\mathrm{coker}(f) = \psi'\,\mathrm{coker}(b)\,\mathrm{coker}(f)$, and so $\mathrm{im}(c)\psi = \psi'\,\mathrm{coker}(b)$ (here $\mathrm{coker}(f)$ is epic). Therefore we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Coker}(f) \arrow[r, two heads, "\psi"] \arrow[d] & \mathrm{Im}(g) \arrow[r, hook] \arrow[d] & Z \arrow[d, "c"] \\
\mathrm{Coker}(f') \arrow[r, two heads, "\psi'"'] & \mathrm{Im}(g') \arrow[r, hook] & Z'
\end{tikzcd}$$

The universal property of kernel induces $\Phi\colon \mathrm{Ker}(\psi) \to \mathrm{Ker}(\psi')$. Since the composition

$$\begin{tikzcd}
\mathrm{Ker}(g) \arrow[r, hook] & Y \arrow[r] & \mathrm{Coker}(f) \arrow[r, "\psi"] & \mathrm{Im}(g) \arrow[r, hook] & Z
\end{tikzcd}$$

is 0 and $\mathrm{im}(g)\colon \mathrm{Im}(g) \to Z$ is monic, the composition

$$\mathrm{Ker}(g) \hookrightarrow Y \longrightarrow \mathrm{Coker}(f) \xrightarrow{\psi} \mathrm{Im}(g)$$

is 0. Then from the universal property of kernel we construct the morphism $\mathrm{Ker}(g) \to \mathrm{Ker}(\psi)$:

$$\begin{tikzcd}
\mathrm{Ker}(g) \arrow[d, dashed] \arrow[dr, "\mathrm{coker}(f)\ker(g)"] \arrow[drr, bend left, "0"] & & \\
\mathrm{Ker}(\psi) \arrow[r] & \mathrm{Coker}(f) \arrow[r, two heads, "\psi"] & \mathrm{Im}(g)
\end{tikzcd}$$

Similarly we have $\mathrm{Ker}(g') \to \mathrm{Ker}(\psi')$. Now consider the diagram

$$\begin{tikzcd}
\mathrm{Ker}(g) \arrow[r] \arrow[d, "\ker(g)"'] & \mathrm{Ker}(\psi) \arrow[r, hook, "\ker(\psi)"] \arrow[d, "\ker(\mathrm{coker}(b))"] & \mathrm{Coker}(f) \arrow[r, two heads, "\psi"] \arrow[d, "\mathrm{coker}(b)"] & \mathrm{Im}(g) \arrow[r] \arrow[d, "\mathrm{im}(c)"] & Z \arrow[d, "c"] \\
\mathrm{Ker}(g') \arrow[r] & \mathrm{Ker}(\psi') \arrow[r, hook, "\ker(\psi')"'] & \mathrm{Coker}(f') \arrow[r, two heads, "\psi'"'] & \mathrm{Im}(g') \arrow[r] & Z'
\end{tikzcd}$$

One may verify that the left square is commutative. Thus the following diagram commutes

$$\begin{tikzcd}
\mathrm{Ker}(g) \arrow[r, two heads] \arrow[d] & \mathrm{Ker}(\psi) \arrow[r, hook] \arrow[d, "\Phi"] & \mathrm{Coker}(f) \arrow[d] \\
\mathrm{Ker}(g') \arrow[r, two heads] & \mathrm{Ker}(\psi') \arrow[r, hook] & \mathrm{Coker}(f')
\end{tikzcd}$$

The uniqueness is obvious from Lemma 1.3. □

Definition 4.17. The **n-th homology** of a complex $C_\bullet$ is defined as

$$H_n(C_\bullet) \coloneqq H\left[C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}\right].$$

We say a chain complex $C_\bullet$ is **acyclic**, if $H_\bullet(C_\bullet) = 0$.

Definition 4.18. A **chain complex map** $u\colon C_\bullet \to D_\bullet$ is a set of maps of morphisms $u_n\colon C_n \to D_n$ s.t. $u_{n-1}d_n = d_{n-1}u_n$, i.e.,

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+2}"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "u_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "u_n"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "u_{n-1}"] & \cdots \\
\cdots \arrow[r, "d'_{n+2}"] & D_{n+1} \arrow[r, "d'_{n+1}"] & D_n \arrow[r, "d'_n"] & D_{n-1} \arrow[r, "d'_{n-1}"] & \cdots
\end{tikzcd}$$

Therefore, the categoty $\mathrm{Kom}(\mathsf{A})$, whose objects are chain complexes and morphisms are chain complex maps, forms a category. It is an abelian category. In this note, we omit the proof.

Now consider the chain complex map $u\colon C_\bullet \to D_\bullet$. From Lemma 4.7, for every $n \in \mathbb{Z}$, there exists $H_n(u)\colon H_n(C) \to H_n(D)$ s.t. the following diagram commutes

$$\begin{tikzcd}
\mathrm{Ker}(d_n) \arrow[r, two heads] \arrow[d] & H_n(C) \arrow[r, hook] \arrow[d, "H_n(u)"] & \mathrm{Coker}(d_{n+1}) \arrow[d] \\
\mathrm{Ker}(d'_n) \arrow[r, two heads] & H_n(D) \arrow[r, hook] & \mathrm{Coker}(d'_{n+1})
\end{tikzcd}$$

Definition 4.19. A chain map $u\colon C_\bullet \to D_\bullet$ is a **quasi-isomorphism** if it induces isomorphisms on all homology:

$$H_n(u)\colon H_n(C) \xrightarrow{\sim} H_n(D), \quad \forall n \in \mathbb{Z}$$

Definition 4.20. Let A be an abelian category, $\mathrm{Kom}(\mathsf{A})$ the category of chain complexes over A. There exists a category $D(\mathsf{A})$ and a functor $Q\colon \mathrm{Kom}(\mathsf{A}) \to D(\mathsf{A})$ with the following properties:

a) $Q(f)$ is an isomorphism for any quasi-isomorphism f.

b) Any functor $F\colon \mathrm{Kom}(A) \to \mathsf{D}$ transforming quasi-isomorphisms into isomorphisms can be uniquely factorized through $D(\mathsf{A})$, i.e., there exists a unique functor $G\colon D(\mathsf{A}) \to \mathsf{D}$ with $F = GQ$.

The category $D(\mathsf{A})$ is called the **derived category** of the abelian category A.

Let B be an arbitrary category and S an arbitrary class of morphisms in B. We show that there exists a universal functor transforming elements of S into isomorphisms. This functor is called the "localization by S" functor $Q\colon \mathsf{B} \to \mathsf{B}\left[S^{-1}\right]$, with the universality property similar to the Definition above.

To do this we set first $\mathrm{Ob}\,\mathsf{B}\left[S^{-1}\right] = \mathrm{Ob}\,\mathsf{B}$ and define Q to be the identity on objects.

To construct morphisms in $\mathsf{B}\left[S^{-1}\right]$ we proceed in several steps.

a) Introduce variables x_s, one for every morphism $s \in S$.

b) Construct an oriented graph Γ as follows:

vertices of Γ = objects of B;

edges of $\Gamma = \{\text{morphisms in } \mathsf{B}\} \cup \{x_s, s \in S\}$;

the edge $X \to Y$ is oriented from X to Y;

the edge x_s has the same vertices as the edge s but the opposite orientation.

c) A **path** in Γ is a finite sequence of edges such that the end of any edge coincides with the beginning of the next one.

d) A morphism in $\mathsf{B}\left[S^{-1}\right]$ is an equivalence class of paths in Γ with the common beginning and the common end. Two paths are equivalent if they can be joined by a chain of elementary equivalences of the following type:

two consecutive arrows in a path can be replaced by their composition;

arrows $X \xrightarrow{s} Y \xrightarrow{x_s} X$ (resp. $Y \xrightarrow{x_s} X \xrightarrow{s} Y$) can be replaced by $X \xrightarrow{\text{id}} X$ (resp. $Y \xrightarrow{\text{id}} Y$).

Finally, the composition of two morphisms is induced by the conjunction of paths and the functor $Q\colon \mathsf{B} \to \mathsf{B}\left[S^{-1}\right]$ maps a morphism $X \to Y$ into the class of corresponding path (of length 1). For any $s \in S$ the morphism $Q(s)$ is clearly an isomorphism in $\mathsf{B}\left[S^{-1}\right]$, the inverse being the class of the path x_s.

Given another functor $F\colon \mathsf{B} \to \mathsf{B}'$ transforming morphisms from S into isomorphisms, the associated functor $G\colon \mathsf{B}\left[S^{-1}\right] \to \mathsf{B}'$ with the condition $F = GQ$ is constructed as follows:

$$G(X) = F(X), \quad X \in \mathrm{Ob}(\mathsf{B}) = \mathrm{Ob}\left(\mathsf{B}\left[S^{-1}\right]\right);$$

$$G(f) = F(f), \quad f \in \mathrm{Mor}(\mathsf{B}),$$

$$G(\text{class of } x_s) = F(s)^{-1}, \quad s \in S.$$

The reader can easily verify that all definitions are unambiguous.

Assume that there exists another $G'\colon \mathsf{B}\left[S^{-1}\right] \to \mathsf{B}'$ satisfies $F = G'Q$, we claim that $G' = G$. Since $\mathrm{Ob}\left(\mathsf{B}\left[S^{-1}\right]\right) = \mathrm{Ob}(\mathsf{B})$, we have $G'(X) = F(X) = G(X)$ for all objects X. Let f be a morphism in B, we have $G'(Q(f)) = F(f)$, so $G'(f) = F(f) = G(f)$. Since $Q(s)$ is an isomorphism in $\mathsf{B}\left[S^{-1}\right]$ with inverse x_s, and functors preserve isomorphisms, $G'(x_s) = G'\left(Q(s)^{-1}\right) = G'(Q(s))^{-1} = F(s)^{-1} = G(X_s)$. Since both functor must preserve composition, they agree on all compositie paths. Therefore $G' = G$ on all morphisms.